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Separation of variables for A_2 Ruijsenaars model and new integral representation for A_2 Macdonald polynomials

V.B. Kuznetsov [1, 2]

Department of Applied Mathematical Studies, University of Leeds,
Leeds LS2 9JT, UK

E.K. Sklyanin [1]

Steklov Mathematical Institute, Fontanka 27, St.Petersburg 191011, Russia

Abstract

Using the Baker-Akhiezer function technique we construct a separation of variables for the classical trigonometric 3-particle Ruijsenaars model (relativistic generalization of Calogero-Moser-Sutherland model). In the quantum case, an integral operator M is constructed from the Askey-Wilson contour integral. The operator M transforms the eigenfunctions of the commuting Hamiltonians (Macdonald polynomials for the root sytem A_2) into the factorized form $S(y_1)S(y_2)$ where $S(y)$ is a Laurent polynomial of one variable expressed in terms of the ${}_3\phi_2(y)$ basic hypergeometric series. The inversion of M produces a new integral representation for the A_2 Macdonald polynomials. We also present some results and conjectures for general n-particle case.

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[1]Part of this work was done while the authors were visiting the Institute of Mathematical Modelling, Technical University of Denmark, Lyngby, Denmark.

[2]On leave from Department of Mathematical and Computational Physics, Institute of Physics, St. Petersburg University, St. Petersburg 198904, Russia.

1. Introduction

The Separation of Variables (SoV) is an approach to quantum integrable systems which can be briefly formulated as follows (for a more detailed discussion see the survey [1]).

Given a quantum-mechanical system of n degrees of freedom possessing n commuting Hamiltonians

$$[H_j, H_k] = 0, \qquad j, k = 1, 2, \ldots, n \tag{1.1}$$

one tries to find an operator M sending any common eigenvector P_λ of the Hamiltonians

$$H_j P_\lambda = h_j P_\lambda \tag{1.2}$$

labelled by the quantum numbers $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ into the product

$$M : P_\lambda \to \prod_{j=1}^{n} S_{\lambda;j}(y_j) \tag{1.3}$$

of functions $S_{\lambda;j}(y_j)$ of one variable each. The original multi-dimensional eigenvalue problem (1.2) is transformed respectively into a set of simpler one-dimensional spectral problems (separated equations)

$$\mathcal{D}_j \left(y_j, \frac{\partial}{\partial y_j}; h_1, \ldots, h_n \right) S_{\lambda;j}(y_j) = 0 \tag{1.4}$$

where $\mathcal{D}_j$ are usually some differential or finite-difference operators in variable y_j depending on the spectral parameters h_k. In the context of the classical Hamiltonian mechanics the above construction corresponds precisely to the standard definition of SoV in the Hamilton-Jacobi equation.

The advent of the Inverse Scattering Method gave new life to SoV providing it with the interpretation of the separated coordinates y_j (in the classical case) as the poles of the Baker-Akhiezer function (properly normalized eigenvector of the corresponding Lax matrix). The unsolved question is, however, how to choose a correct normalization of B-A function to obtain SoV for a given Lax matrix. Nevertheless, as an heuristic recipe, the above idea has proved to be quite efficient and allowed to find SoV for a few new classes of classical integrables systems. In particular, SoV was found for the systems arising from the r-matrices satisfying the classical Yang-Baxter equation in case of A_{n-1} (sl_n) Lie algebra. In the cases $n = 2$ and $n = 3$ the construction of SoV has been successfully transferred to the quantum case (see [1] and references therein).

Pursuing the goal to extend the applicability of the B-A function recipe, in our previous paper [2] we have studied the A_2 Calogero-Sutherland model which does not fall into the previously studied cases since it posesses a dynamical (non-numeric) r-matrix [3]. In the quantum case, our construction of SoV has produced a new integral representation for the eigenfunctions of the A_2 C-S Hamiltonians (known as Jack polynomials) in terms of ${}_3F_2$ hypergeometric polynomials.

In the present paper we generalize the results of [2] to the 3-particle Ruijsenaars model [4] which is a relativistic analog of the C-S model. The corresponding eigenfunctions (Macdonald polynomials [5, 6]) are q-analogs of Jack polynomials. No surprize that the corresponding separated functions are Laurent polynomials expressed in terms of ${}_3\phi_2$ basic hypergeometric series. We present also some results and conjectures for the general n-particle problem, for instance, we connect the A_{n-1} type basic hypergeometric separation polynomials $S_\lambda(y)$ to a terminated case of the ϕ_D type q-Lauricella function of $n-1$ variables.

The paper is organized as follows. In Section 2 we describe the classical Ruijsenaars model and, using B-A function technique, construct a SoV. Though the results of this section are not used directly in what follows, they provide a useful background for subsequent treatment of the quantum case. In Section 3 the standard facts concerning the quantum Ruijsenaars model and Macdonald polynomials are collected. In Section 4, after introducing the quantum Hamiltonians and Macdonald polynomials, we describe the integral operator M performing a SoV and formulate the main theorem whose proof takes the rest of the section and part of the next one. The main part of the proof is contained in Section 4 where the properties of the operator M are studied, whereas in Section 5 the results concerning the separated equation (certain 3-rd order q-difference equation and its n-th order generalization), as well as its polynomial solutions, are collected. The main technical tool allowing us to study the operator M is the famous Askey-Wilson integral identity (A.15).

Generally, SoV is aimed to simplify the multidimensional spectral problem by reducing it to a series of one-dimensional ones. In case of the Calogero-Sutherland and Ruijsenaars models, however, the spectrum and eigenfunctions are well known and studied by independent means. The main benefit of SoV in application to these models is rather producing new relations between special functions. In particular, inverting the operator M one obtains a new integral representation for A_2 Macdonald polynomials in terms of ${}_3\phi_2$ basic hypergeometric functions, which is done in the end of Section 4. In section 6 we discuss the obtained results and the possibility of their generalization to A_n, $n > 2$ case. Two Appendices, A and B, contain, respectively, a collection of necessary formulas from q-analysis and some auxiliary results concerning operator M.

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2. Classical Ruijsenaars model

In the spirit of q-analysis, we prefer to use exponentiated canonical coordinates and momenta.

Definition 1. *The variables* (X_j, x_j) $j = 1, \ldots, n$ *on a* $2n$*-dimensional symplectic manifold form a* Weyl canonical system *if they possess the Poisson brackets*

$$\{X_j, X_k\} = \{x_j, x_k\} = 0, \quad \{X_j, x_k\} = -iX_j x_k \delta_{jk}, \qquad j, k = 1, \ldots, n \tag{2.1}$$

or, equivalently, the symplectic form ω *is expressed as* $\omega = i \sum_j d\ln X_j \wedge d\ln x_j = d(i \sum_j \ln X_j d\ln x_j)$.

The n-particle (A_{n-1}) trigonometric Ruijsenaars model [4] is formulated in terms of the Weyl canonical system (T_j, t_j) where $|t_j| = 1$, $T_j \in \mathbb{R}$ $(j = 1, 2, \ldots, n)$. The Hamiltonians H_i are defined as

$$H_i = \sum_{\substack{J \subset \{1,\ldots,n\} \\ |J| = i}} \left(\prod_{\substack{j \in J \\ k \in \{1,\ldots,n\} \setminus J}} v_{jk} \right) \left(\prod_{j \in J} T_j \right), \qquad i = 1, \ldots, n, \tag{2.2}$$

where

$$v_{jk} = \frac{\ell^{-\frac{1}{2}} t_j - \ell^{\frac{1}{2}} t_k}{t_j - t_k}, \qquad \ell \in (1, \infty). \tag{2.3}$$

Proposition 1 [4, 7]

The Hamiltonians H_j *Poisson commute.*

$$\{H_j, H_k\} = 0, \qquad j, k = 1, \ldots, n. \tag{2.4}$$

Define the Lax matrix (L operator) by the formula

$$L(u) = D(u)E(u) \tag{2.5}$$

where

$$D_{jk} = \frac{(\ell - 1)(1 - \ell^n u)}{2\ell^{\frac{n+1}{2}}(1 - u)} \left(\prod_{i \neq j} v_{ji} \right) T_j \delta_{jk}, \tag{2.6}$$

$$E_{jk} = \frac{1 + \ell^n u}{1 - \ell^n u} - \frac{t_j + \ell t_k}{t_j - \ell t_k}. \tag{2.7}$$

Proposition 2 [4]

The characteristic polynomial of the matrix $L(u)$ *(2.5) generates the Hamiltonians (2.2)*

$$\begin{aligned} &(-1)^n \ell^{\frac{n(n-1)}{2}} (1 - \ell^n u)(1 - u)^n \det(z - L(u)) \\ &\quad = \sum_{k=0}^{n} (-1)^k \ell^{\frac{n-1}{2}k} (1 - \ell^k u)(1 - u)^k (1 - \ell^n u)^{n-k} H_{n-k} z^k \end{aligned} \tag{2.8}$$

where we assume $H_0 \equiv 1$.

In the 3-particle (A_2) case which we consider henceforth we have, respectively,

$$H_1 = v_{12}v_{13}T_1 + v_{21}v_{23}T_2 + v_{31}v_{32}T_3, \tag{2.9a}$$

$$H_2 = v_{13}v_{23}T_1T_2 + v_{12}v_{32}T_1T_3 + v_{21}v_{31}T_2T_3, \tag{2.9b}$$

$$H_3 = T_1T_2T_3, \tag{2.9c}$$

$$D = \frac{(\ell-1)(1-\ell^3 u)}{2\ell^2(1-u)} \operatorname{diag}\{v_{12}v_{13}T_1, v_{21}v_{23}T_2, v_{31}v_{32}T_3\}, \tag{2.10}$$

$$E_{jk} = \frac{1+\ell^3 u}{1-\ell^3 u} - \frac{t_j + \ell t_k}{t_j - \ell t_k}, \tag{2.11}$$

and

$$\begin{aligned} \ell^3(1-u)^2 \det(z - L(u)) = z^3\ell^3(1-u)^2 - z^2\ell^2(1-u)(1-\ell^2 u)H_1 \\ + z\ell(1-\ell u)(1-\ell^3 u)H_2 - (1-\ell^3 u)^2 H_3. \end{aligned} \tag{2.12}$$

To find a SoV for the Ruijsenaars system we use the recipe discussed in the Introduction and choose for the separated coordinates y_j the poles upon u of the Baker-Akhiezer function $\psi(u)$ (an eigenvector of $L(u)$) normalized by the condition that its 3-rd component $\psi_3(u)$ is constant. The canonically conjugated (in the Weyl sense) variables Y_j are chosen as the eigenvalues of $L(y_j)$. For the detailed discussion of the B-A function recipe see [1] though the construction described below is quite self-contained.

Define two functions $A_1(u)$ and $A_2(u)$ by the formulas

$$A_k(u) := L_{kk} - \frac{L_{3k}L_{k,3-k}}{L_{3,3-k}} = T_k\alpha_k(u), \qquad k = 1, 2 \tag{2.13}$$

$$\alpha_k(u) := \frac{(1-\ell^3 u)(\ell t_3 u - t_{3-k})(t_k - \ell t_3)}{\ell(1-u)(\ell^2 t_3 u - t_{3-k})(\ell t_k - t_3)}, \qquad k = 1, 2\,. \tag{2.14}$$

The separated variables y_j are defined from the equation

$$A_1(y) = A_2(y). \tag{2.15}$$

It is easy to see that (2.15) has 3 solutions one of which $y = \ell^{-3}$ we ignore since it is a constant. The remaining two roots we denote y_1 and y_2. From the easily verified invariance of $\alpha_1(u)/\alpha_2(u)$ under the transformation $u \mapsto u^{-1}t_1t_2t_3^{-2}\ell^{-\frac{3}{2}}$ it follows that

$$y_1 y_2 = \frac{t_1 t_2}{t_3^2 \ell^3}. \tag{2.16}$$

The conjugated variables Y_j are defined as

$$Y_j = A_1(y_j) = A_2(y_j), \qquad j = 1, 2. \tag{2.17}$$

Equivalently, the four variables Y_1, Y_2, y_1, y_2 are defined through four equations

$$Y_j = T_k\alpha_k(y_j), \qquad j, k \in \{1, 2\}. \tag{2.18}$$

Theorem 1 *The variables Y_j, y_j satisfy the separated equations*

$$Y_j^3\ell^3(1-y_j)^2 - Y_j^2\ell^2(1-y_j)(1-\ell^2 y_j)H_1 \\ +Y_j\ell(1-\ell y_j)(1-\ell^3 y_j)H_2 - (1-\ell^3 y_j)^2 H_3 = 0, \qquad j=1,2 \tag{2.19}$$

which, by virtue of (2.12), imply that $\det(Y_j - L(y_j)) = 0$.

Proof. Substitute into (2.19) the expressions (2.9) for H_j and split the left-hand-side of (2.19) into two terms

$$T_3 Z_1 + Y_j Z_2 = 0 \tag{2.20}$$

where

$$Z_1 = -(1-y_j)(1-\ell^2 y_j)\ell^2 v_{31}v_{32}Y_j^2 \\ +(1-\ell y_j)(1-\ell^3 y_j)\ell(v_{12}v_{32}T_1Y_j + v_{21}v_{31}T_2Y_j) - (1-\ell^3 y_j)^2 T_1T_2, \tag{2.21a}$$

$$Z_2 = (1-y_j)^2\ell^3 Y_j^2 - (1-y_j)(1-\ell^2 y_j)\ell^2(v_{12}v_{13}T_1Y_j + v_{21}v_{23}T_2Y_j) \\ +(1-\ell y_j)(1-\ell^3 y_j)v_{13}v_{23}T_1T_2. \tag{2.21b}$$

To prove (2.19) it is sufficient to show that $Z_1 = Z_2 = 0$. Replacing Y_j in (2.21) by $T_1\alpha_1(y_j)$ or $T_2\alpha_2(y_j)$ in such a way that the factor T_1T_2 could be cancelled from $Z_{1,2}$ we obtain that $Z_{1,2} = 0$ follows from two algebraic identities for $\alpha_{1,2}$

$$-(1-y)(1-\ell^2 y)\ell^2 v_{31}v_{32}\alpha_1(y)\alpha_2(y) \\ +(1-\ell y)(1-\ell^3 y)\ell(v_{12}v_{32}\alpha_2(y) + v_{21}v_{31}\alpha_1(y)) - (1-\ell^3 y)^2 = 0, \tag{2.22a}$$

$$(1-y)^2\ell^3\alpha_1(y)\alpha_2(y) - (1-y)(1-\ell^2 y)\ell^2(v_{12}v_{13}\alpha_2(y) + v_{21}v_{23}\alpha_1(y)) \\ +(1-\ell y)(1-\ell^3 y)\ell v_{13}v_{23} = 0, \tag{2.22b}$$

which are verified directly. ∎

The third pair of separated variables is defined as

$$x := t_3, \qquad X := T_1T_2T_3, \tag{2.23}$$

the corresponding separated equation being

$$X - H_3 = 0. \tag{2.24}$$

Theorem 2 *The variables $(X, Y_1, Y_2; x, y_1, y_2)$ form a Weyl canonical system in the sense of the definition 1.*

Proof. Let us introduce new variables:

$$t_+ = t_1^{1/2} t_2^{1/2} t_3^{-1}, \qquad t_- = t_1^{1/2} t_2^{-1/2}, \tag{2.25a}$$

$$T_+ = T_1 T_2, \qquad T_- = T_1 T_2^{-1}, \tag{2.25b}$$

and also

$$y_+ = y_1^{1/2} y_2^{1/2}, \qquad y_- = y_1^{1/2} y_2^{-1/2}, \tag{2.26a}$$

$$Y_+ = Y_1 Y_2, \qquad Y_- = Y_1 Y_2^{-1}. \tag{2.26b}$$

Obviously, $(X, T_-, T_+; x, t_-, t_+)$ is also a Weyl canonical system. Note that

$$y_+ = t_+ \ell^{-\frac{3}{2}} \tag{2.27}$$

because of (2.16). Note also that from (2.14) it follows that $Y_\pm$, $y_\pm$ depend only on $T_\pm$, $t_\pm$ and do not contain X, x.

It remains to show that the transformation from $(T_-, T_+; t_-, t_+)$ to $(Y_-, Y_+; y_-, y_+)$ is canonical that is $(Y_-, Y_+; y_-, y_+)$ is again a Weyl canonical system. To this end, it suffices to construct the generating function $F(Y_+, y_-; t_+, t_-)$ of the canonical transformation such that [8]

$$i \ln T_\pm = t_\pm \frac{\partial F}{\partial t_\pm}, \qquad i \ln Y_- = -y_- \frac{\partial F}{\partial y_-}, \qquad i \ln y_+ = Y_+ \frac{\partial F}{\partial Y_+}. \tag{2.28}$$

and $d(F - i \ln Y_+ \ln y_+) = i(\ln T_- d \ln t_- + \ln T_+ d \ln t_+) - i(\ln Y_- d \ln y_- + \ln Y_+ d \ln y_+)$. Recalling the definition of the Euler dilogarithm [9]

$$\mathrm{Li}_2(z) := -\int_0^z \frac{dt}{t} \ln(1-t) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{2.29}$$

and introducing the notation

$$\mathcal{L}(\nu; x, y) := \mathrm{Li}_2(\nu x y) + \mathrm{Li}_2(\nu x y^{-1}) + \mathrm{Li}_2(\nu x^{-1} y) + \mathrm{Li}_2(\nu x^{-1} y^{-1}) \tag{2.30}$$

we define $F := i \ln Y_+ \ln(\ell^{-\frac{3}{2}} t_+) + \widetilde{F}$,

$$\begin{aligned} \widetilde{F} \;&:=\; i(\mathcal{L}(\ell^{-\frac{1}{2}}; y_-, t_-) + \mathcal{L}(\ell^{-1}; t_+, t_-) - \mathcal{L}(\ell^{-\frac{3}{2}}; t_+, y_-) \\ &\quad - \mathrm{Li}_2(t_-^2) - \mathrm{Li}_2(t_-^{-2})). \end{aligned} \tag{2.31}$$

It is a matter of direct calculation to verify, using (2.18) and (2.27), that F satisfies (2.28). ■

The identities (2.19) and (2.24) and canonicity of the variables $(X, Y_1, Y_2; x, y_1, y_2)$ established above provide, by definition [1], a SoV for the A_2 Ruijsenaars system.

3. Quantization

We collect here the standard facts concerning the quantum n-particle (A_{n-1}) Ruijsenaars model [4, 7] and the corresponding Macdonald polynomials [5, 6].

Throughout the paper $\mathbb{Z}$ stands for the set of integers, the notations $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{\leq 0}$ are self-evident.

The quantum Ruijsenaars model is described in terms of the multiplication and shift operators, resp. t_j and T_j ($j = 1, \ldots, n$) acting on functions of t_j

$$(t_j f)(\vec{t}) := t_j f(\vec{t}), \qquad (T_j f)(\vec{t}) := f(\ldots, q t_j, \ldots) \tag{3.1}$$

(we do not make distinction between variables and operators t_j). Here q is the quantum deformation parameter related to the Planck constant $\hbar > 0$ as

$$q = e^{-\hbar}, \qquad q \in (0, 1). \tag{3.2}$$

The operators T_j, t_j satisfy the Weyl commutation relations

$$[T_j, T_k] = [t_j, t_k] = 0, \qquad T_j t_k = \begin{cases} q t_k T_j, & j = k \\ t_k T_j, & j \neq k \end{cases} \tag{3.3}$$

which produce the Poisson brackets (2.1) in the classical limit $\hbar \to 0$ by the standard correspondence rule $[,] = -i\hbar\{,\} + O(\hbar^2)$.

The commuting quantum Hamiltonians H_j

$$[H_j, H_k] = 0, \qquad j, k = 1, \ldots, n \tag{3.4}$$

are given by the same formulas (2.2) as in the classical case with the fixed operator ordering (T_j to the right). We assume that

$$\ell = q^{-g} = e^{g\hbar}, \qquad g > 0, \quad \ell \in (1, \infty) \tag{3.5}$$

(note that both in the classical and nonrelativistic limits $\hbar \to 0$, $q = e^{-\hbar} \to 1$ but in the classical limit $g \to \infty$, $\ell = \text{const}$ whereas in the nonrelativistic limit $g = \text{const}$, $\ell \to 1$).

The operators H_k leave invariant the space $\mathrm{Sym}(t_1, \ldots, t_n)$ of symmetric Laurent polynomials in variables t_j. A basis in $\mathrm{Sym}(t_1, \ldots, t_n)$ is given by the monomial symmetric functions m_λ labelled by the sequences $\lambda = \{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n\}$ of integers $\lambda_j \in \mathbb{Z}$ (dominant weights) and expressed as $m_\lambda = \sum t_1^{\nu_1} \ldots t_n^{\nu_n}$ where the sum is taken over all distinct permutations ν of λ.

Denote $|\lambda| \equiv \sum_{j=1}^n \lambda_j$. The dominant order on the dominant weights λ is defined as

$$\lambda' \preceq \lambda \quad \Longleftrightarrow \quad \left\{ |\lambda'| = |\lambda|\,; \quad \sum_{j=k}^n \lambda'_j \leq \sum_{j=k}^n \lambda_j, \quad k = 2, \ldots, n \right\}. \tag{3.6}$$

The Macdonald polynomials $P_\lambda^{(\ell;q)} \in \mathrm{Sym}(t_1, \ldots, t_n)$ are uniquely defined as joint eigenvectors of H_k in $\mathrm{Sym}(t_1, \ldots, t_n)$

$$H_k P_\lambda^{(\ell;q)} = h_k P_\lambda^{(\ell;q)} \tag{3.7}$$

labelled by the dominant weight λ and normalized by the condition

$$P_\lambda^{(\ell;q)} = \sum_{\lambda' \preceq \lambda} \kappa_{\lambda\lambda'} m_{\lambda'}, \qquad \kappa_{\lambda\lambda} = 1. \tag{3.8}$$

The corresponding eigenvalues h_k are

$$h_k = \sum_{j_1 < \ldots < j_k} \mu_{j_1} \ldots \mu_{j_k}, \qquad \mu_j = q^{\lambda_j} \ell^{\frac{n+1}{2} - j}. \tag{3.9}$$

Note that our parameter ℓ and parameter t used in [5, 6] relate as $\ell = t^{-1}$.

The polynomials $P_\lambda^{(\ell;q)}$ are orthogonal

$$\frac{1}{(2\pi i)^n} \oint\limits_{|t_1|=1} \frac{dt_1}{t_1} \cdots \oint\limits_{|t_n|=1} \frac{dt_n}{t_n} \bar{P}_\lambda^{(\ell;q)}(\vec{t}) P_{\lambda'}^{(\ell;q)}(\vec{t}) \Delta(\vec{t}) = 0, \qquad \lambda \neq \lambda' \tag{3.10}$$

with respect to the weight

$$\Delta(t_1, \ldots, t_n) = \prod_{j \neq k} \frac{(t_j t_k^{-1}; q)_\infty}{(\ell^{-1} t_j t_k^{-1}; q)_\infty} \tag{3.11}$$

(see (A.2) for the notation).

In the limit $\hbar \to 0$, $g =$ const the appropriate linear combinations of H_k produce the Hamiltonians of the nonrelativistic Calogero-Sutherland model, and the Macdonald polynomials go over into the Jack polynomials, see [2].

In the present paper we consider only the simplest nontrivial case $n = 3$.

The Hamiltonians H_k being given by (2.9), the formulas (3.9) produce, respectively,

$$h_1 = \ell q^{\lambda_1} + q^{\lambda_2} + \ell^{-1} q^{\lambda_3}, \quad h_2 = \ell q^{\lambda_1 + \lambda_2} + q^{\lambda_1 + \lambda_3} + \ell^{-1} q^{\lambda_2 + \lambda_3}, \quad h_3 = q^{|\lambda|} \tag{3.12}$$

for their eigenvalues labelled by the ordered triplets $\{\lambda_1 \leq \lambda_2 \leq \lambda_3\} \in \mathbb{Z}^3$.

For instance,

$$m_{000} = 1, \quad m_{001} = t_1 + t_2 + t_3, \quad m_{011} = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad m_{002} = t_1^2 + t_2^2 + t_3^2,$$

$$m_{111} = t_1 t_2 t_3, \quad m_{012} = t_1 t_2^2 + t_1^2 t_2 + t_1 t_3^2 + t_1^2 t_3 + t_2 t_3^2 + t_2^2 t_3,$$

$$m_{112} = t_1^2 t_2 t_3 + t_1 t_2^2 t_3 + t_1 t_2 t_3^2, \quad m_{022} = t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2, \quad m_{003} = t_1^3 + t_2^3 + t_3^3.$$

$$P_{000}^{(\ell;q)} = m_{000}, \quad P_{001}^{(\ell;q)} = m_{001}, \quad P_{011}^{(\ell;q)} = m_{011}, \quad P_{002}^{(\ell;q)} = m_{002} + \tfrac{(1-\ell)(1+q)}{q-\ell} m_{011},$$

$$P_{111}^{(\ell;q)} = m_{111}, \quad P_{012}^{(\ell;q)} = m_{012} + \tfrac{(1-\ell)(q(2+\ell)+1+2\ell)}{q-\ell^2} m_{111},$$

$$P_{112}^{(\ell;q)} = m_{112}, \quad P_{022}^{(\ell;q)} = m_{022} + \tfrac{(1-\ell)(1+q)}{q-\ell} m_{112},$$

$$P_{003}^{(\ell;q)} = m_{003} + \tfrac{(1-\ell)(1+q+q^2)}{q^2-\ell} m_{012} + \tfrac{(1-\ell)^2(1+q)(1+q+q^2)}{(q-\ell)(q^2-\ell)} m_{111}.$$

4. Operator M

We are now going to describe the integral operator M (1.3) producing the SoV. Generally speaking, the kernel $\mathcal{M}$ of M should depend on 6 variables: $\mathcal{M}(x, y_1, y_2 \mid t_1, t_2, t_3)$. However, by analogy with the classical case (section 2) and the nonrelativistic limit [2], it is natural to assume that $\mathcal{M}$ contains two δ-functions corresponding to the constraints $x = t_3$ (2.23) and, respectively, (2.16). There remains thus only one integration in M. Again by analogy with the previously studied cases, the kernel $\mathcal{M}$ is most conveniently described in terms of the variables $t_\pm$ (2.25a) and $y_\pm$ (2.26a).

So, let us introduce the operator M

$$\begin{aligned} &M : \Psi(t_1, t_2, t_3) \to \Phi(x, y_1, y_2) \\ &= \frac{1}{2\pi i} \int\limits_{\Gamma^{t_+,y_-}_{g,2g}} \frac{dt_-}{t_-} \mathcal{M}((y_1 y_2)^{\frac{1}{2}}, (y_1/y_2)^{\frac{1}{2}} \,|\, t_-) \Psi(\ell^{\frac{3}{2}} x (y_1 y_2)^{\frac{1}{2}} t_-, \ell^{\frac{3}{2}} x (y_1 y_2)^{\frac{1}{2}} t_-^{-1}, x) \end{aligned} \tag{4.1}$$

with the kernel

$$\mathcal{M}(y_+, y_- \mid t_-) = \frac{(1-q)(q;q)^2_\infty (t_-^2, t_-^{-2}; q)_\infty \, \mathcal{L}_q\Big(\ell^{-\frac{3}{2}}; y_-, y_+ \ell^{\frac{3}{2}}\Big)}{2B_q(g, 2g)\, \mathcal{L}_q\Big(\ell^{-\frac{1}{2}}; y_-, t_-\Big)\, \mathcal{L}_q\Big(\ell^{-1}; t_-, y_+ \ell^{\frac{3}{2}}\Big)} \tag{4.2}$$

where the notation (A.7) and (B.2) is used. For the definition of the cycle $\Gamma^{t_+,y_-}_{g,2g}$ which depends on g, $y_{1,2}$ see (B.4) and (A.16).

Remark. In the classical limit, as $q \to 1$, $\ell = \text{const}$, using (A.20) and $\ln \mathcal{L}_q(\nu; x, y) \sim -\hbar^{-1} \mathcal{L}(\nu; x, y)$ one obtains that the asymptotics $\ln \mathcal{M} \sim -i\hbar^{-1} \widetilde{F}$ of the kernel $\mathcal{M}$ is determined by the regular part $\widetilde{F}$ (2.31) of the generating function of the canonical transformation producing classical SoV. As for the nonrelativistic limit, $\hbar \to 0$, $g = \text{const}$, the easiest way to reproduce the results of [2] is to compare the action of the operators M and its nonrelativistic analog on polynomials, see theorem 4.

Now we are in a position to formulate our main result.

Theorem 3 *The operator M (4.1) transforms any A_2 Macdonald polynomial $P^{(\ell;q)}_\lambda(t_1, t_2, t_3)$ into the product*

$$M : P^{(\ell;q)}_\lambda(t_1, t_2, t_3) \to c_\lambda x^{|\lambda|} S^{(\ell;q)}_\lambda(y_1) S^{(\ell;q)}_\lambda(y_2) \tag{4.3}$$

of functions of one variable only, where the Laurent polynomials $S^{(\ell;q)}_{\lambda_1\lambda_2\lambda_3}$

$$S^{(\ell;q)}_\lambda(y) = \sum_{k=\lambda_1}^{\lambda_3} \chi^{(\ell;q)}_{\lambda,k} y^k \tag{4.4}$$

are expressed in terms of the basic hypergeometric series (A.9)

$$S^{(\ell;q)}_\lambda(y) = y^{\lambda_1} (y;q)_{1-3g}\, {}_3\phi_2 \left[\begin{matrix} \ell^3 q^{1-\lambda_{31}}, \ell^2 q^{1-\lambda_{21}}, \ell q \\ \ell^2 q^{1-\lambda_{31}}, \ell q^{1-\lambda_{21}} \end{matrix} ; q, y \right], \tag{4.5}$$

where $\lambda_{jk} \equiv \lambda_j - \lambda_k$. The coefficients $\chi_{\lambda,k}^{(\ell;q)}$ are given by

$$\chi_{\lambda,k}^{(\ell;q)} = (q\ell^3)^{k-\lambda_1} \frac{(q^{-1}\ell^{-3};q)_{k-\lambda_1}}{(q;q)_{k-\lambda_1}} {}_4\phi_3 \left[\begin{matrix} q^{\lambda_1-k}, \ell^3 q^{1-\lambda_{31}}, \ell^2 q^{1-\lambda_{21}}, \ell q \\ \ell^3 q^{\lambda_1-k+2}, \ell^2 q^{1-\lambda_{31}}, \ell q^{1-\lambda_{21}} \end{matrix} ; q, q \right]. \tag{4.6}$$

The normalization coefficient c_λ equals

$$c_\lambda = \ell^{4\lambda_1-\lambda_2} \frac{(\ell^{-2};q)_{\lambda_{31}} (\ell^{-2};q)_{\lambda_{32}} (\ell^{-1};q)_{\lambda_{21}}}{(\ell^{-3};q)_{\lambda_{31}} (\ell^{-1};q)_{\lambda_{32}} (\ell^{-2};q)_{\lambda_{21}}}. \tag{4.7}$$

The proof of the above result will occupy the rest of this section and a part of the next one. Our proof parallels the similar one for the nonrelativistic Calogero-Sutherland model [2].

We begin with proving the factorization (4.3) of $MP_\lambda^{(\ell;q)}$. The first step is to show that the image $MP_\lambda^{(\ell;q)}$ satisfies certain q-difference equations in x, y_1, y_2. Let us introduce the operators Y_j ($j = 1, 2$) acting on functions of y_k as (cf. (3.1))

$$(Y_j f)(\vec{y}) = f(\ldots, qy_j, \ldots). \tag{4.8}$$

Using $y_\pm = (y_1 y_2^{\pm 1})^{1/2}$ (2.26a) one can write also

$$(Y_1 f)(y_+, y_-) = f(q^{\frac{1}{2}} y_+, q^{\frac{1}{2}} y_-), \qquad (Y_2 f)(y_+, y_-) = f(q^{\frac{1}{2}} y_+, q^{-\frac{1}{2}} y_-). \tag{4.9}$$

Similarly,

$$(T_1 f)(t_+, t_-) = f(q^{\frac{1}{2}} t_+, q^{\frac{1}{2}} t_-), \qquad (T_2 f)(t_+, t_-) = f(q^{\frac{1}{2}} t_+, q^{-\frac{1}{2}} t_-). \tag{4.10}$$

We define also the operator X as $X(f)(x) = f(qx)$.

Let us introduce the operator expression $\mathcal{D}$

$$\begin{aligned} \mathcal{D}(u, z; \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3) := {} & (1-qu)(1-q^2u)\ell^3 z^3 - (1-qu)(1-q^2\ell^2 u)\ell^2 z^2 \mathcal{H}_1 \\ & +(1-q\ell u)(1-q^2\ell^3 u)\ell z \mathcal{H}_2 - (1-q\ell^3 u)(1-q^2\ell^3 u)\mathcal{H}_3 \end{aligned} \tag{4.11}$$

which can be considered as a quantum generalization of the characteristic polynomial (2.12). The ordering is important in (4.11) since we are going to replace the parameters u, z, $\mathcal{H}_j$ by non-commuting operators.

Proposition 3 *The operator M (4.1) satisfies the equations*

$$XM - MH_3 = 0, \tag{4.12}$$

$$\mathcal{D}(y_j, Y_j; MH_1, MH_2, MH_3) = 0, \qquad j = 1, 2 \tag{4.13}$$

where $H_{1,2,3}$ are the quantum Hamiltonians (2.9).

Proof. Though the equality (4.12) is easy to derive from the fact that M respects the constraint $x = t_3$, or directly from (4.1), we shall proceed, however, in a more methodical fashion allowing to prove both (4.12) and (4.13) in the same way. Let us rewrite first the operator identities (4.12) and (4.13) for M as algebraic identities for the kernel $\mathcal{M}$ (4.2).

We define the Lagrange adjoint Hamiltonians H_k^* as

$$H_1^* = T_1^{-1} v_{12} v_{13} + T_2^{-1} v_{21} v_{23} + T_3^{-1} v_{31} v_{32}, \tag{4.14a}$$

$$H_2^* = T_1^{-1} T_2^{-1} v_{13} v_{23} + T_1^{-1} T_3^{-1} v_{12} v_{32} + T_2^{-1} T_3^{-1} v_{21} v_{31}, \tag{4.14b}$$

$$H_3^* = T_1^{-1} T_2^{-1} T_3^{-1}, \tag{4.14c}$$

$$\frac{1}{2\pi i} \oint \frac{dt}{t} \, f(t)(Hg)(t) = \frac{1}{2\pi i} \oint \frac{dt}{t} \, (H^* f)(t) g(t). \tag{4.15}$$

In particular, $T_j^* = T_j^{-1}$. Considering M in (4.12) and (4.13) as integral operator, we can use integration by parts and switch H_k to the kernel $\mathcal{M}$ replacing them by H_k^* according to (4.15) which results in the q-difference equations for $\mathcal{M}$:

$$(X - H_3^*)\mathcal{M} = 0, \tag{4.16}$$

$$\mathcal{D}(y_j, Y_j; H_1^*, H_2^*, H_3^*)\mathcal{M} = 0, \quad j = 1, 2. \tag{4.17}$$

While (4.16) is obvious, (4.17) needs more consideration. Note that, by virtue of (4.9) and (4.10), the action of $\mathcal{D}$ on $\mathcal{M}(y_+, y_- \mid t_-)$ is well defined. Note also that the equations (4.16) and (4.17) are the quantum counterparts, resp., of the classical separated equations (2.24) and (2.19).

The next step is to notice that the kernel $\mathcal{M}$ (4.2) satisfies the four first order q-difference equations

$$Y_j T_k \mathcal{M} = \breve{\alpha}_k(y_j)\mathcal{M}, \qquad j, k \in \{1, 2\}, \tag{4.18}$$

where (compare to classical (2.14))

$$\breve{\alpha}_k(y) = \frac{(1 - q\ell^3 y)(t_k - \ell t_3)(\ell t_3 y - t_{3-k})(q t_k - t_{3-k})}{\ell(1 - y)(q\ell t_k - t_3)(q\ell^2 t_3 y - t_{3-k})(t_k - t_{3-k})}, \qquad k = 1, 2 \tag{4.19}$$

which are verified directly from (4.2) using the relations (A.4). Note that (4.18) is the quantum counterpart of (2.17)–(2.18).

Remark. It is easy to verify that the system (4.18) is holonomic, that is the operators $\breve{\alpha}_k(y_j)^{-1} Y_j T_k$ commute, provided y_j and t_k are bound by (2.16).

We proceed now to derive the third-order q-difference relations in y_j (4.17) for $\mathcal{M}$ from the first-order relations (4.18). The proof parallels that of theorem 1 for the classical case. Let us write down the equations (4.17) explicitely

$$\Big[(1 - q y_j)(1 - q^2 y_j)\ell^3 Y_j^3 - (1 - q y_j)(1 - q^2 \ell^2 y_j)\ell^2 Y_j^2 H_1^* \\ + (1 - q\ell y_j)(1 - q^2 \ell^3 y_j)\ell Y_j H_2^* - (1 - q\ell^3 y_j)(1 - q^2 \ell^3 y_j) H_3^*\Big] \mathcal{M} = 0, \tag{4.20}$$

then substitute into (4.20) the expressions (4.14) for H_j^* and split the left-hand-side of (4.20) into two terms

$$T_3^{-1}\check{Z}_1 + Y_j\check{Z}_2 = 0 \tag{4.21}$$

where

$$\check{Z}_1 = \Big[-(1-qy_j)(1-q^2\ell^2y_j)\ell^2Y_j^2v_{31}v_{32} - (1-q\ell^3y_j)(1-q^2\ell^3y_j)T_1^{-1}T_2^{-1}$$

$$+(1-q\ell y_j)(1-q^2\ell^3y_j)\ell Y_j(T_1^{-1}v_{12}v_{32} + T_2^{-1}v_{21}v_{31})\Big]\,\mathcal{M}, \tag{4.22a}$$

$$\check{Z}_2 = \Big[(1-y_j)(1-qy_j)\ell^3Y_j^2 + (1-\ell y_j)(1-q\ell^3y_j)\ell T_1^{-1}T_2^{-1}v_{13}v_{23}$$

$$-(1-y_j)(1-q\ell^2y_j)\ell^2Y_j(T_1^{-1}v_{12}v_{13} + T_2^{-1}v_{21}v_{23})\Big]\,\mathcal{M}. \tag{4.22b}$$

Introducing the notation

$$\check{\alpha}_{12}(y) \equiv \check{\alpha}_1(qy)|_{t_2:=qt_2}\check{\alpha}_2(y) = \check{\alpha}_2(qy)|_{t_1:=qt_1}\check{\alpha}_1(y)\,, \tag{4.23}$$

$$\check{v}_{jk} = \frac{\ell^{-\frac{1}{2}}t_j - q\ell^{\frac{1}{2}}t_k}{t_j - qt_k} \tag{4.24}$$

and noting that

$$T_kv_{jk} = \check{v}_{jk}T_k, \qquad v_{jk}T_j = T_j\check{v}_{jk}, \tag{4.25}$$

it is easy to verify the algebraic identities for $\check{\alpha}_{1,2}$

$$-(1-qy)(1-q^2\ell^2y)\ell^2\check{v}_{31}\check{v}_{32}\check{\alpha}_{12}(y) - (1-q\ell^3y)(1-q^2\ell^3y)$$

$$+(1-q\ell y)(1-q^2\ell^3y)\ell(\check{v}_{12}\check{v}_{32}\check{\alpha}_2(y) + \check{v}_{21}\check{v}_{31}\check{\alpha}_1(y)) = 0, \tag{4.26a}$$

$$(1-y)(1-qy)\ell^3\check{\alpha}_{12}(y) + (1-\ell y)(1-q\ell^3y)\ell v_{13}v_{23}$$

$$-(1-y)(1-q\ell^2y)\ell^2(\check{v}_{12}v_{13}\check{\alpha}_2(y) + \check{v}_{21}v_{23}\check{\alpha}_1(y)) = 0. \tag{4.26b}$$

Now we insert $T_kT_k^{-1}$ in appropriate places in (4.22) in such a way that $T_1^{-1}T_2^{-1}$ could be carried out to the left of $[\ldots]$. Then we push the products YT to the right using (4.25) until they hit $\mathcal{M}$, so that (4.18) could be applied. The equalities $\check{Z}_1 = 0$, $\check{Z}_2 = 0$ and therefore (4.20) and (4.13) follow then immediately from (4.26). ■

Proposition 4 *The function $(MP_\lambda^{(\ell;q)})(x, y_1, y_2)$ satisfies the q-difference equations (separated equations)*

$$(X - h_3)MP_\lambda^{(\ell;q)} = 0, \tag{4.27}$$

$$\mathcal{D}(y_j, Y_j; h_1, h_2, h_3)MP_\lambda^{(\ell;q)} = 0, \qquad j = 1, 2. \tag{4.28}$$

Proof. Apply the operator expressions (4.12) and (4.13) to the function $P_\lambda^{(\ell;q)}$. Using the operator ordering convention and the fact that Macdonald polynomials $P_\lambda^{(\ell;q)}$ are the eigenfunctions of the Hamiltonians H_j (3.7) one replaces H_j by h_j. Since h_j are just numbers, the operator M can be applied then directly to $P_\lambda^{(\ell;q)}$ which results in (4.27) and (4.28). ∎

In order to derive the factorization (4.3) of $MP_\lambda^{(\ell;q)}$ we need more specific information about how M acts on the symmetric polynomials from $\mathrm{Sym}(t_1, t_2, t_3)$. Note that solutions to (4.18), as to any q-difference equations, are defined only up to a factor invariant under q-shifts (quasiconstant). Our choice (4.2) of the kernel $\mathcal{M}$ corresponds to a particular choice of the quasiconstant which is crucial for the results given below.

Since the kernel $\mathcal{M}$ (4.2) is a particular case (B.8) of the kernel $\mathcal{M}_{\alpha\beta}$ (B.7), we can make use of the results obtained for $\mathcal{M}_{\alpha\beta}$ in Appendix B.

Let us define few polynomial spaces. Let $\mathrm{Sym}(t_1, t_2, t_3)$ be the space of Laurent polynomials symmetric w.r.t. permutations of 3 variables t_1, t_2, t_3. A basis in $\mathrm{Sym}(t_1, t_2, t_3)$ is given by m_λ or $P_\lambda^{(\ell;q)}$. Let $\mathrm{Sym}(t_1, t_2; t_3)$ be the space of Laurent polynomials of the same 3 variables, symmetric only w.r.t. $t_1 \leftrightarrow t_2$. Obviously, $\mathrm{Sym}(t_1, t_2; t_3) \supset \mathrm{Sym}(t_1, t_2, t_3)$. Though the Macdonald polynomials belong to $\mathrm{Sym}(t_1, t_2, t_3)$ it is convenient to define M on a larger space $\mathrm{Sym}(t_1, t_2; t_3)$.

Let $\mathrm{Ref}(t_-; t_+; t_3)$ be the space of Laurent polynomials in $t_\pm$, t_3 which are reflexive in t_- (invariant w.r.t. $t_- \to t_-^{-1}$) and even in $t_\pm$ (invariant w.r.t. $(t_-, t_+) \to (-t_-, -t_+)$. Note that the change of variables $(t_1, t_2, t_3) \to (t_-, t_+, t_3)$, see (2.25a), provides an isomorphism $\mathrm{Sym}(t_1, t_2; t_3) \simeq \mathrm{Ref}(t_-; t_+; t_3)$.

The spaces $\mathrm{Sym}(y_1, y_2; x) \simeq \mathrm{Ref}(y_-; y_+; x)$ are defined similarly.

Proposition 5

$$M : \mathrm{Sym}(t_1, t_2; t_3) \to \mathrm{Sym}(y_1, y_2; x).$$

In particular, the image of a Macdonald polynomial $P_\lambda^{(\ell;q)} \in \mathrm{Sym}(t_1, t_2, t_3)$ *also lies in* $\mathrm{Sym}(y_1, y_2; x)$.

Proof. The proposition 13 from Appendix B implies that $M : \mathrm{Ref}(t_-; t_+; t_3) \to \mathrm{Ref}(y_-; y_+; x)$. Using the isomorphisms $\mathrm{Sym}(t_1, t_2; t_3) \simeq \mathrm{Ref}(t_-; t_+; t_3)$ and $\mathrm{Sym}(y_1, y_2; x) \simeq \mathrm{Ref}(y_-; y_+; x)$ we conclude the proof. ∎

Now everything is ready to prove the main statement of the theorem 3.

Proposition 6 *The operator* M *transforms any Macdonald polynomial* $P_\lambda^{(\ell;q)}$ *into the product (4.3).*

Proof. We have already established that $MP_\lambda^{(\ell;q)}$ is a Laurent polynomial (proposition 5) satisfying the q-difference equations (4.27) and (4.28). The factorization (4.3) follows from the fact that x^{h_3} and $S_\lambda^{(\ell;q)}(y)$ are the unique, up to a constant factor, Laurent-polynomial solutions, of the q-difference equations, resp. $(X - h_3)f(x) = 0$, and $\mathcal{D}(y, Y; h_1, h_2, h_3)f(y) = 0$. The first statement is obvious, as for the second one, see proposition 12. ∎

Though for the theorem 3 we have used only the polynomiality of $MP_\lambda^{(\ell;q)}$, in fact, the action of M on $\mathrm{Sym}(t_1, t_2; t_3)$ can be described in much more detail. Namely, taking the formula (B.13) from Appendix B, making the substitutions (B.8) and performing the changes of variables (2.25a) and (2.26a) one obtains the following result.

Theorem 4 *Consider the basis in* $\mathrm{Sym}(t_1, t_2; t_3)$

$$p_{jk\nu} := t_3^{j-2k} t_1^k t_2^k (\ell^{-1} t_1 t_3^{-1}, \ell^{-1} t_2 t_3^{-1}; q)_\nu, \qquad j, k \in \mathbb{Z}, \quad \nu \in \mathbb{Z}_{\geq 0}, \tag{4.29}$$

and in $\mathrm{Sym}(y_1, y_2; x)$

$$\widetilde{p}_{jk\nu} := x^j y_1^k y_2^k (y_1, y_2; q)_\nu, \qquad j, k \in \mathbb{Z}, \quad \nu \in \mathbb{Z}_{\geq 0}, \tag{4.30}$$

respectively. The operator M *acts on* $p_{jk\nu}$ *as follows*

$$M : p_{jk\nu} \to \ell^{3k} \frac{(\ell^{-2}; q)_\nu}{(\ell^{-3}; q)_\nu} \widetilde{p}_{jk\nu}. \tag{4.31}$$

Postponing the proof of the formulas (4.5) and (4.6) for the next section, we can prove now the final statement of theorem 3.

Proposition 7 *The normalization coefficient* c_λ *in (4.3) is given by (4.7).*

Proof. In this case, it is convenient to make use of the isomorphisms described above and to think of M as acting from $\mathrm{Ref}(t_-; t_+; t_3)$ into $\mathrm{Ref}(y_-; y_+; x)$. Comparing the asymptotics of the monomial symmetric functions m_λ

$$m_{\lambda_1\lambda_2\lambda_3} \sim t_-^{\lambda_3-\lambda_1} t_+^{\lambda_3+\lambda_1} t_3^{\lambda_1+\lambda_2+\lambda_3}, \qquad t_- \to \infty$$

and of the polynomial $p_{jk\nu}$ (4.29)

$$\begin{aligned} p_{jk\nu} &\equiv t_3^j t_+^{2k} (\ell^{-1} t_+ t_-, \ell^{-1} t_+ t_-^{-1})_\nu \\ &\sim (-1)^\nu q^{\frac{\nu(\nu-1)}{2}} \ell^{-\nu} t_3^j t_+^{2k+\nu} t_-^\nu, \qquad t_- \to \infty \end{aligned}$$

we conclude that the transition matrix between the bases m_λ and $p_{jk\nu}$ is triangular

$$m_\lambda = (-1)^{\lambda_{31}} q^{-\frac{\lambda_{31}(\lambda_{31}-1)}{2}} \ell^{\lambda_{31}} p_{|\lambda|,\lambda_1,\lambda_{31}} + \sum_{\nu<\lambda_{31}} \sum_{j,k} (\ldots) p_{jk\nu}. \tag{4.32}$$

Given the mutual triangularity (3.8) of the bases $P_\lambda^{(\ell;q)}$ and m_λ, it means that the expansion of $P_\lambda^{(\ell;q)}$ in $p_{jk\nu}$ has the same structure as (4.32). Using then (4.31) and the asymptotics of $\widetilde{p}_{jk\nu}$ (4.30)

$$\widetilde{p}_{jk\nu} \equiv x^j y_+^{2k} (y_+ y_-, y_+ y_-^{-1})_\nu \sim (-1)^\nu q^{\frac{\nu(\nu-1)}{2}} x^j y_+^{2k+\nu} y_-^\nu, \qquad y_- \to \infty$$

we obtain

$$
\begin{aligned}
MP_\lambda^{(\ell;q)} &= (-1)^{\lambda_{31}} q^{-\frac{\lambda_{31}(\lambda_{31}-1)}{2}} \ell^{2\lambda_1+\lambda_3} \frac{(\ell^{-2};q)_{\lambda_{31}}}{(\ell^{-3};q)_{\lambda_{31}}} \widetilde{p}_{|\lambda|,\lambda_1,\lambda_{31}} + \dots \\
&\sim \ell^{2\lambda_1+\lambda_3} \frac{(\ell^{-2};q)_{\lambda_{31}}}{(\ell^{-3};q)_{\lambda_{31}}} x^{|\lambda|} y_+^{\lambda_3+\lambda_1} y_-^{\lambda_{31}}, \qquad y_- \to \infty.
\end{aligned} \tag{4.33}
$$

On the other hand, (4.4) implies that

$$
c_\lambda \, x^{|\lambda|} \, S_\lambda^{(\ell;q)}(y_+ y_-) \, S_\lambda^{(\ell;q)}(y_+ y_-^{-1}) \sim c_\lambda \, \chi_{\lambda,\lambda_1}^{(\ell;q)} \, \chi_{\lambda,\lambda_3}^{(\ell;q)} \, x^{|\lambda|} y_+^{\lambda_3+\lambda_1} y_-^{\lambda_3-\lambda_1}
$$

whence

$$
c_\lambda \, \chi_{\lambda,\lambda_1}^{(\ell;q)} \, \chi_{\lambda,\lambda_3}^{(\ell;q)} = \frac{(\ell^{-2};q)_{\lambda_3-\lambda_1}}{(\ell^{-3};q)_{\lambda_3-\lambda_1}} \ell^{\lambda_3+2\lambda_1}. \tag{4.34}
$$

It remains only to use the formulas (5.14) proved in the next section, and obtain (4.7). ■

Compared to [2] our formula (4.7) for the normalization coefficients c_λ is new, and its nonrelativistic analog

$$
c_\lambda = \frac{(2g)_{\lambda_{31}}(2g)_{\lambda_{32}}(g)_{\lambda_{21}}}{(3g)_{\lambda_{31}}(g)_{\lambda_{32}}(2g)_{\lambda_{21}}}, \qquad (\alpha)_k \equiv \alpha(\alpha+1)\dots(\alpha+k-1),
$$

fills the gap in the description given in [2] of the integral representation for Jack polynomials analogous to (4.38).

We conclude this section with a list of results concerning the inverse operator M^{-1}. All the preparatory work being done in Appendix B, it remains only to use the correspondence (B.8) between $M_{\alpha\beta}$ and M.

From (B.20) and (B.21) it follows that M^{-1} is an integral operator

$$
\begin{aligned}
&M^{-1} : \Phi(x,y_1,y_2) \to \Psi(t_1,t_2,t_3) \\
&= \frac{1}{2\pi i} \int_{\Gamma_{-g,3g}^{t_+,t_-}} \frac{dy_-}{y_-} \widetilde{\mathcal{M}}\left(\frac{(t_1t_2)^{\frac12}}{t_3}, \left(\frac{t_1}{t_2}\right)^{\frac12} \Big| \, y_- \right) \Phi\left(t_3, \frac{\ell^{-\frac32}(t_1t_2)^{\frac12} y_-}{t_3}, \frac{\ell^{-\frac32}(t_1t_2)^{\frac12}}{t_3 y_-} \right)
\end{aligned} \tag{4.35}
$$

with the kernel

$$
\widetilde{\mathcal{M}}(t_+,t_- \mid y_-) = \frac{(1-q)(q;q)_\infty^2 (y_-^2, y_-^{-2};q)_\infty \, \mathcal{L}_q(\ell^{-1}; t_-, t_+)}{2B_q(-g,3g)\, \mathcal{L}_q\Big(\ell^{\frac12}; y_-, t_-\Big)\, \mathcal{L}_q\Big(\ell^{-\frac32}; y_-, t_+\Big)}. \tag{4.36}
$$

Reversing (4.31) one obtains the formula for the action of M^{-1} on the basis $\widetilde{p}_{jk\nu}$

$$
M^{-1} : \widetilde{p}_{jk\nu} \to \ell^{-3k} \frac{(\ell^{-3};q)_\nu}{(\ell^{-2};q)_\nu} p_{jk\nu}. \tag{4.37}
$$

Reversing (4.3) provides a new integral representation of A_2 Macdonald polynomials in terms of the Laurent polynomials $S_\lambda^{(\ell;q)}(y)$ (4.5)

$$
M^{-1} : c_\lambda x^{|\lambda|} S_\lambda^{(\ell;q)}(y_1) S_\lambda^{(\ell;q)}(y_2) \to P_\lambda^{(\ell;q)}(t_1,t_2,t_3). \tag{4.38}
$$

Finally, from the propositions 14 and 15 it follows that for positive integer g the operator M^{-1} turns into a q-difference operator of order g:

$$M^{-1}:\Phi(x,y_1,y_2)\to\sum_{k=1}^{g}\xi_k\left(\frac{(t_1t_2)^{\frac{1}{2}}}{t_3},\left(\frac{t_1}{t_2}\right)^{\frac{1}{2}}\right)\Phi\Big(t_3,q^{g+k}\frac{t_1}{t_3},q^{2g-k}\frac{t_2}{t_3}\Big) \tag{4.39}$$

where $\xi_k(r,s)$ is given by (B.15). The result is not surprising in view of the similar result for the nonrelativistic case [2] where M^{-1} becomes a differential operator of order g for $g\in\mathbb{Z}_{\geq 0}$. In [2] this result was derived using a representation of M^{-1} in terms of the fractional differentiation operator. In the relativistic case it is also possible to relate M^{-1} with a sort of fractional q-difference operator. We intend to touch this subject in a separate paper.

5. Separated equation

In this section the results are collected concerning the Laurent polynomials $S_\lambda^{(\ell;q)}(y)$ and the corresponding q-difference equations. Since all the results are easy to generalize from $n=3$ to arbitrary n, we give them in the most general form.

Conjecture 1 *The correct generalization of the formula (4.5) for $S_\lambda^{(\ell;q)}(y)$ for any n is given by*

$$S_\lambda^{(\ell;q)}(y)=y^{\lambda_1}(y;q)_{1-ng}\,{}_n\phi_{n-1}\left[\begin{matrix}a_1,\dots,a_n\\ b_1,\dots,b_{n-1}\end{matrix};q,y\right] \tag{5.1}$$

where

$$a_j=\ell^{n-j+1}q^{\lambda_1-\lambda_{n-j+1}+1},\qquad b_j=a_j\ell^{-1}. \tag{5.2}$$

Proposition 8 *$S_\lambda^{(\ell;q)}(y)$ is a Laurent polynomial in y of the form*

$$S_\lambda^{(\ell;q)}(y)=\sum_{k=\lambda_1}^{\lambda_n}\chi_{\lambda,k}^{(\ell;q)}y^k. \tag{5.3}$$

Proof. Observe, first, that if $a=bq^\nu$ for some positive integer ν then

$$\frac{(a;q)_k}{(b;q)_k}=\frac{(bq^k;q)_\nu}{(b;q)_\nu} \tag{5.4}$$

is a polynomial in q^k of degree ν whose coefficients are rational functions in b and q. As a consequence, if $a_{j+1}=b_jq^{\nu_j}$ then

$$P_N(q^k)\equiv\frac{(a_2;q)_k\dots(a_n;q)_k}{(b_1;q)_k\dots(b_{n-1};q)_k}=\frac{(b_1q^k;q)_{\nu_1}\dots(b_{n-1}q^k;q)_{\nu_{n-1}}}{(b_1;q)_{\nu_1}\dots(b_{n-1};q)_{\nu_{n-1}}} \tag{5.5}$$

is a polynomial in q^k of degree $N = \nu_1 + \cdots + \nu_{n-1}$.

In our case, $\nu_j = \lambda_{n-j+1} - \lambda_{n-j}$, $N = \lambda_n - \lambda_1$ by virtue of (5.2), and from (5.1) and (A.9) one obtains

$$ {}_n\phi_{n-1}\left[\begin{matrix} a_1, \ldots, a_n \\ b_1, \ldots, b_{n-1} \end{matrix} ; q, y\right] = \sum_{k=0}^{\infty} \frac{(a_1;q)_k}{(q;q)_k}\, y^k P_N(q^k) \tag{5.6} $$

where $P_N(q^k)$ is given by (5.5). It remains now to apply the following lemma.

Lemma 1 *Let $P_N(y)$ be a polynomial in y of degree $\le N$. Then*

$$ \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, y^k P_N(q^k) = Q_N(y) \frac{(aq^N y;q)_\infty}{(y;q)_\infty} \tag{5.7} $$

where $Q_N(y)$ is a polynomial in y of degree $\le N$.

Proof. It is sufficient to consider the polynomials $P_N(q^k) = (q^{k-\nu+1};q)_\nu$ for $\nu = 0, 1, \ldots, N$ forming a basis in the polynomial ring. Then

$$ \begin{aligned} \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, y^k (q^{k-\nu+1};q)_\nu &= \sum_{k=\nu}^{\infty} [\cdots] = \sum_{k=\nu}^{\infty} \frac{(a;q)_k}{(q;q)_{k-\nu}}\, y^k \\ &= \sum_{k=0}^{\infty} \frac{(a;q)_{k+\nu}}{(q;q)_k}\, y^{k+\nu} = (a;q)_\nu y^\nu \sum_{k=0}^{\infty} \frac{(aq^\nu;q)_k}{(q;q)_k}\, y^k. \end{aligned} \tag{5.8} $$

Using the formula (A.11) and the identity $(aq^\nu;q)_\infty = (aq^\nu;q)_{N-\nu}(aq^N;q)_\infty$ one obtains finally the expression (5.7) where $Q_N(y) = (a;q)_\nu y^\nu (aq^\nu y;q)_{N-\nu}$. ■

Applying the above lemma to the case of the polynomial $P_N(q^k)$ given by (5.6) and $a = a_1 = \ell^n q^{\lambda_1 - \lambda_n + 1}$ we obtain finally that $y^{-\lambda_1} S_\lambda^{(\ell;q)}(y)$ is a polynomial of degree $\le \lambda_n - \lambda_1$. ■

Proposition 9 *The coefficients $\chi_{\lambda,k}^{(\ell;q)}$ in the expansion (5.3) are given by*

$$ \chi_{\lambda,k}^{(\ell;q)} = (q\ell^n)^{k-\lambda_1} \frac{(q^{-1}\ell^{-n};q)_{k-\lambda_1}}{(q;q)_{k-\lambda_1}}\, {}_{n+1}\phi_n\left[\begin{matrix} q^{\lambda_1-k}, a_1, \ldots, a_n \\ q^{\lambda_1-k+2}\ell^n, b_1, \ldots, b_{n-1} \end{matrix} ; q, q\right]. \tag{5.9} $$

In particular, for $n = 3$, (5.9) produces (4.6).

Proof. We know already that $S_\lambda^{(\ell;q)}(y)$ is a Laurent polynomial and is thus defined for any $y \in \mathbb{C} \setminus \{0, \infty\}$. Suppose for a while that $|y| < 1$. Then both factors $(y;q)_{1-ng}$ and ${}_n\phi_{n-1}$ in (5.1) are given by the convergent series (A.11) and (A.9), respectively. Multiplying the two power series in y we observe that the coefficients at y^k is expressed in terms of ${}_{n+1}\phi_n$ series:

$$ S_\lambda^{(\ell;q)}(y) = y^{\lambda_1} \sum_{k=0}^{\infty} (q\ell^n)^k \frac{(q^{-1}\ell^{-N};q)_k}{(q;q)_k}\, {}_{n+1}\phi_n\left[\begin{matrix} q^{-k}, a_1, \ldots, a_n \\ q^{-k+2}\ell^n, b_1, \ldots, b_{n-1} \end{matrix} ; q, q\right] y^k. \tag{5.10} $$

In fact, the sum in (5.10) is finite: $\sum_{k=0}^{\lambda_{n1}}$. To see this, use the formula (1.9.11) from [10]: let ν, $k_1,\dots,k_n\in\mathbb{Z}_{\geq 0}$, then

$$
{}_{n+1}\phi_n\left[\begin{matrix} q^{-\nu}, b_1q^{k_1},\dots,b_nq^{k_n} \\ b_1,\dots,b_n\end{matrix}\; ;q,q\right]=0 \tag{5.11}
$$

for $\nu>k_1+\cdots+k_n$. Substituting

$$
\nu=k\,,\qquad b_n=q^{2-k}\ell^n\,,\qquad k_n=\lambda_1-\lambda_n+k-1\,,
$$

$$
b_j=\ell^{n-j}q^{\lambda_1-\lambda_{n-j+1}+1}\,,\qquad k_j=\lambda_{n-j+1}-\lambda_{n-j},\qquad j=1,\dots,n-1\,,
$$

we obtain that

$$
{}_{n+1}\phi_n\left[\begin{matrix} q^{-k}, a_1,\dots,a_n \\ q^{2-k}\ell^n, b_1,\dots,b_{n-1}\end{matrix}\; ;q,q\right]=0
$$

for $k\geq\lambda_n-\lambda_1+1$, hence the sum in (5.10) is finite: $\sum_{k=0}^{\lambda_{n1}}$. The coefficient at y^k in (5.10) produces, respectively, (5.9). ■

For the sake of reference we present a short list of polynomials $S_\lambda^{(\ell;q)}(y)$ in case $n=3$:

$$
S_{000}^{(\ell;q)}=1,\quad S_{001}^{(\ell;q)}=1+\tfrac{\ell^2y}{\ell+1},\quad S_{011}^{(\ell;q)}=1+\ell\;(\ell+1)\,y,
$$

$$
S_{002}^{(\ell;q)}=1+\tfrac{\ell^2(q\ell+\ell-q-1)y}{\ell^2-q}+\tfrac{(\ell-q)\ell^4y^2}{(\ell^2-q)(\ell+1)}\,,
$$

$$
S_{012}^{(\ell;q)}=1+\tfrac{(\ell^3+\ell^2q+\ell^2-\ell q-\ell-q)\ell y}{\ell^2-q}+\ell^3y^2\,,
$$

$$
S_{022}^{(\ell;q)}=1+\tfrac{(\ell^2-1)(q+1)\ell y}{\ell-q}+\tfrac{(\ell^2-q)(\ell+1)\ell^2y^2}{\ell-q}\,,
$$

$$
S_{003}^{(\ell;q)}=1+\tfrac{(1+q+q^2)(\ell-1)\ell^2y}{\ell^2-q^2}+\tfrac{(1+q+q^2)(\ell-1)\ell^4y^2}{(\ell+q)(\ell^2-q)}+\tfrac{(\ell-q^2)\ell^6y^3}{(\ell+1)(\ell+q)(\ell^2-q)}\,,
$$

$$
S_{013}^{(\ell;q)}=1+\tfrac{(\ell^3+q^2\ell^2+\ell^2q+\ell^2-q^2\ell-\ell q-\ell-q^2)\ell y}{\ell^2-q^2}
$$

$$
+\tfrac{(q\ell^3+\ell^3+\ell^2q^2+\ell^2q+\ell^2-q^3\ell-\ell q^2-\ell q-q^3-q^2)(\ell-1)\ell^3y^2}{(\ell^2-q)(\ell^2-q^2)}+\tfrac{(\ell-q)\ell^5y^3}{\ell^2-q}\,,
$$

$$
S_{023}^{(\ell;q)}=1+\tfrac{(q\ell^3+\ell^3+\ell^2q^2+\ell^2q+\ell^2-q^3\ell-q^2\ell-\ell q-q^3-q^2)(\ell-1)\ell y}{(\ell-q)^2(\ell+q)}
$$

$$
+\tfrac{(\ell^3+\ell^2q^2+\ell^2q+\ell^2-q^2\ell-\ell q-\ell-q^2)(\ell^2-q)\ell^2y^2}{(\ell-q)^2(\ell+q)}+\tfrac{(\ell^2-q)\ell^4y^3}{\ell-q}\,,
$$

$$
\begin{aligned}
S_{033}^{(\ell;q)} &= 1+\tfrac{(1+q+q^2)(\ell^2-1)\ell y}{\ell-q^2}+\tfrac{(1+q+q^2)(\ell^2-1)(\ell^2-q)\ell^2y^2}{(\ell-q)(\ell-q^2)}\\
&\quad+\tfrac{(\ell+1)(\ell+q)(\ell^2-q)\ell^3y^3}{\ell-q^2}\,.
\end{aligned}
$$

Remark. It easy to give more simple expressions for some of $\chi_{\lambda,k}^{(\ell;q)}$ such as

$$
\chi_{\lambda,\lambda_1}^{(\ell;q)}=1, \tag{5.12}
$$

and

$$
\chi_{\lambda,\lambda_n}^{(\ell;q)}=\ell^{|\lambda|-n\lambda_1}\prod_{j=1}^{n-1}\frac{(\ell^{-j};q)_{\lambda_j-\lambda_1}(\ell^{-j};q)_{\lambda_n-\lambda_{n-j}}}{(\ell^{-j};q)_{\lambda_{j+1}-\lambda_1}(\ell^{-j};q)_{\lambda_n-\lambda_{n-j+1}}}. \tag{5.13}
$$

which for $n=3$ produce

$$\chi^{(\ell;q)}_{\lambda,\lambda_1}=1, \qquad \chi^{(\ell;q)}_{\lambda,\lambda_3}=\ell^{-2\lambda_1+\lambda_2+\lambda_3}\frac{(\ell^{-1};q)_{\lambda_{32}}(\ell^{-2};q)_{\lambda_{21}}}{(\ell^{-2};q)_{\lambda_{32}}(\ell^{-1};q)_{\lambda_{21}}}. \tag{5.14}$$

The formula (5.12) is obvious. To obtain (5.13), use the summation formula (1.9.10) from [10]:

$$\begin{aligned}&{}_{n+1}\phi_n\left[\begin{matrix} q^{-\nu},\beta,\beta_1 q^{k_1},\ldots,\beta_{n-1}q^{k_{n-1}} \\ \beta q,\beta_1,\ldots,\beta_{n-1}\end{matrix};q,q\right]\\ &\qquad =\frac{(q;q)_\nu(\beta_1/\beta;q)_{k_1}\ldots(\beta_{n-1}/\beta;q)_{k_{n-1}}}{(\beta q;q)_\nu(\beta_1;q)_{k_1}\ldots(\beta_{n-1};q)_{k_{n-1}}}\,\beta^\nu,\end{aligned} \tag{5.15}$$

where $\nu,k_1,\ldots,k_{n-1}\in\mathbb{Z}_{\ge 0}$ and $\nu\ge k_1+\cdots+k_{n-1}$. Substituting

$$\nu=\lambda_{n1}, \qquad \beta=\ell^n q^{1-\lambda_{n1}}\equiv a_1\,,$$

$$\beta_j=\ell^j q^{1-\lambda_{j+1}+\lambda_1}\equiv b_{n-j}, \qquad k_j=\lambda_{j+1}-\lambda_j, \quad j=1,\ldots,n-1\,,$$

we obtain, after a series of equivalent transformations (see Appendix I to [10]), the expression (5.13).

Remark. There is also a simple formula for $S^{(\ell;q)}_\lambda(\ell^{-n})$:

$$S^{(\ell;q)}_\lambda(\ell^{-n})=\ell^{-n\lambda_1}(\ell^{-n};q)_{\lambda_{n1}}\prod_{j=1}^{n-1}\frac{(\ell^{-j};q)_{\lambda_j-\lambda_1}}{(\ell^{-j};q)_{\lambda_{j+1}-\lambda_1}}, \tag{5.16}$$

or, for $n=3$,

$$S^{(\ell;q)}_\lambda(\ell^{-3})=\ell^{-3\lambda_1}\frac{(\ell^{-2};q)_{\lambda_{21}}(\ell^{-3};q)_{\lambda_{31}}}{(\ell^{-1};q)_{\lambda_{21}}(\ell^{-2};q)_{\lambda_{31}}}, \tag{5.17}$$

which are proved in a way similar to (5.13) using the formula (1.9.9) from [10].

The polynomials $S^{(\ell;q)}_\lambda(y)$ can be expressed also in terms of the q-Lauricella function (A.12).

Proposition 10 *The following equalities hold:*

$$\begin{aligned} S^{(\ell;q)}_\lambda(y) &= y^{\lambda_1}\frac{(q\ell^n q^{\lambda_{1n}}y;q)_{\lambda_{n1}}}{\prod_{j=1}^{n-1}(q^{\lambda_1-\lambda_{n-j+1}+1}\ell^{n-j};q)_{\lambda_{n-j+1}-\lambda_{n-j}}}\\ &\quad\times\phi_D\left[\begin{matrix} a';b'_1,\ldots,b'_{n-1} \\ c\end{matrix};q;x_1,\ldots,x_{n-1}\right], \end{aligned} \tag{5.18}$$

where $\lambda_{ij}\equiv\lambda_i-\lambda_j$ *and*

$$a'=y, \quad c=q\ell^n q^{\lambda_{1n}}y, \quad x_j=q\ell^{n-j}q^{\lambda_1-\lambda_{n-j}}, \quad b'_j=q^{\lambda_{n-j}-\lambda_{n-j+1}}, \tag{5.19}$$

for $j=1,\ldots,n-1$. *Another expression for* $S^{(\ell;q)}_\lambda(y)$ *reads*

$$S^{(\ell;q)}_\lambda(y)=y^{\lambda_1}(q\ell y;q)_{(1-n)g}\left(\prod_{i=1}^{n-1}(a_i;q)_g\right)\ \phi_D\left[\begin{matrix} y;\ell^{-1},\ldots,\ell^{-1} \\ q\ell y\end{matrix};q;a_1,\ldots,a_{n-1}\right]. \tag{5.20}$$

Proof. The formula (5.18) is obtained by substituting the parameters (5.19) into Andrews' formula (A.13) for q-Lauricella function and comparing the result to (5.1). Note that $c/a' \equiv a_1$, $x_j \equiv a_{j+1}$, $b'_j x_j \equiv b_j$.

Similarly, substituting into (A.13) the parameters $a' = y$, $c = q\ell y$, $b'_j = \ell^{-1}$, $x_j = a_j$ $(j = 1, \ldots, n-1)$ such that $c/a' \equiv a_n$, $b'_j x_j \equiv b_j$, one arrives at (5.20). ■

Corollary 1. Substituting the definition (A.12) of ϕ_D into formula (5.18) we obtain another explicit representation for $S_\lambda^{(\ell;q)}(y)$:

$$S_\lambda^{(\ell;q)}(y) = y^{\lambda_1} \frac{1}{\prod_{j=1}^{n-1} (q^{\lambda_1 - \lambda_{n-j+1}+1} \ell^{n-j}; q)_{\lambda_{n-j+1}-\lambda_{n-j}}}$$

$$\times \sum_{k_1=0}^{\lambda_n - \lambda_{n-1}} \cdots \sum_{k_{n-1}=0}^{\lambda_2-\lambda_1} (q\ell^n q^{\lambda_{1n}+k_1+\ldots+k_{n-1}} y; q)_{\lambda_{n1}-k_1-\ldots-k_{n-1}} \, (y;q)_{k_1+\ldots+k_{n-1}}$$

$$\times \prod_{j=1}^{n-1} \frac{(q^{\lambda_{n-j}-\lambda_{n-j+1}}; q)_{k_j} (q\ell^{n-j} q^{\lambda_1 - \lambda_{n-j}})^{k_j}}{(q;q)_{k_j}} \,. \tag{5.21}$$

Corollary 2. It is possible also to represent $S_\lambda^{(\ell;q)}(y)$ as a q-integral (A.8):

$$S_\lambda^{(\ell;q)}(q^x) = q^{\lambda_1 x} \frac{(q\ell^n q^{\lambda_{1n}} q^x; q)_{\lambda_{n1}}}{\prod_{j=1}^{n-1} (q^{\lambda_1-\lambda_{n-j+1}+1} \ell^{n-j}; q)_{\lambda_{n-j+1}-\lambda_{n-j}}}$$

$$\times \frac{1}{B_q(x, \lambda_{1n}+1-ng)} \int_0^1 d_q t \; t^{x-1} \frac{(tq;q)_{\lambda_{1n}-ng}}{\prod_{j=1}^{n-1} (tq\ell^{n-j} q^{\lambda_1-\lambda_{n-j}}; q)_{\lambda_{n-j}-\lambda_{n-j+1}}} \,. \tag{5.22}$$

To obtain the formula (5.22) rewrite Andrews' formula (A.13) as a q-integral

$$\phi_D \left[\begin{matrix} q^\alpha; q^{\beta_1}, \ldots, q^{\beta_{n-1}} \\ q^\gamma \end{matrix} ; q; x_1, \ldots, x_{n-1} \right] = \frac{1}{B_q(\alpha, \gamma-\alpha)} \int_0^1 d_q t \; t^{\alpha-1} \frac{(tq;q)_{\gamma-\alpha-1}}{\prod_{j=1}^{n-1} (tx_j; q)_{\beta_j}} \,. \tag{5.23}$$

and substitute

$$\alpha = x \quad (y = q^x), \quad \gamma = \lambda_{1n} + 1 - ng + x, \quad \beta_j = \lambda_{n-j} - \lambda_{n-j+1}, \quad x_j = q\ell^{n-j} q^{\lambda_1 - \lambda_{n-j}} \,.$$

The rest of the results are concerned with the separated q-difference equations for the polynomials $S_\lambda^{(\ell;q)}(y)$.

Proposition 11 *The polynomial $f(y) := S_\lambda^{(\ell;q)}(y)$ (5.1) satisfies the q-difference equation*

$$\sum_{k=0}^{n} (-1)^k \ell^{\frac{n-1}{2}k} (1 - q^k \ell^k y)(y;q)_k \, (q^{k+1}\ell^n y; q)_{n-k} \, h_{n-k} \, f(q^k y) = 0 \tag{5.24}$$

where, h_k are given by (3.9) and, as in the classical case (2.8), we assume $h_0 \equiv 1$.

Proof. Using the definitions (5.1) and (5.2) together with (3.9), it is a matter of straightforward calculation to transform the q-difference equation (A.10) for ${}_n\phi_{n-1}$ into (5.24). ∎

In fact, the factor $(1-q^n\ell^n y)$ can be cancelled from (5.24) which results in the equation

$$\begin{aligned}&(-1)^n\ell^{\frac{n(n-1)}{2}}(y;q)_n h_0 f(q^n y)\\&+\sum_{k=0}^{n-1}(-1)^k\ell^{\frac{n-1}{2}k}(1-q^k\ell^k y)(y;q)_k\,(q^{k+1}\ell^n y;q)_{n-k-1}\,h_{n-k}\,f(q^k y)=0\,.\end{aligned}\tag{5.25}$$

In the case $n=3$ the q-difference equation (5.24) takes the form

$$\mathcal{D}(y,Y;h_1,h_2,h_3)f(y)=0$$

where $\mathcal{D}$ is given by (4.11), or, explicitely,

$$\begin{aligned}&(1-qy)(1-q^2y)\ell^3\,f(q^3y)-(1-qy)(1-q^2\ell^2y)\ell^2h_1\,f(q^2y)\\&+(1-q\ell y)(1-q^2\ell^3y)\ell h_2\,f(qy)-(1-q\ell^3y)(1-q^2\ell^3y)h_3\,f(y)=0.\end{aligned}\tag{5.26}$$

Proposition 12 *Let*

$$G_n^{(0)}:=\mathbb{Z}\cup\frac{1}{2}\mathbb{Z}\cup\ldots\cup\frac{1}{n-1}\mathbb{Z},\qquad G_n^{(1)}:=\left\{\frac{1}{n},\frac{2}{n},\ldots,\frac{n-2}{n}\right\}.$$

Then, for all $g>0$ except for the finite number of points $g\in G_n\equiv G_n^{(0)}\cap G_n^{(1)}$, the separated polynomial $f(y):=S_\lambda^{(\ell;q)}(y)$ (5.1) is the only, up to a constant factor, Laurent-polynomial solution to the q-difference equation (5.24).

In particular, $G_3=\emptyset$, so for $n=3$ the uniqueness of L.-p. solution holds $\forall g>0$.

Proof. In the nonrelativistic case [2] the analog of the equation (5.24) is a differential equation having 3 regular singularities: $0,1,\infty$, and the uniqueness of L.-p. solution is proved by analysis of the corresponding characteristic exponents. As shown below, the argument can be translated rather directly to the q-difference case.

Let $f(y)$ be a non-zero Laurent-polynomial solution to (5.24) or, equivalently, (5.25). Then, subsituting into (5.25) the values $y=q^{-j}$, $j=0,1,2,\ldots$, one observes that $f(q^{-j})$ can be determined recursively, starting from $f(1)$ since the factor $(y;q)_k$ cuts away the terms with $k>j$. The only obstacle could be the vanishing of the factor $(q^{k+1}\ell^n y;q)_{n-k-1}$ for $k=0$ which may happen only for $g\in G_n^{(1)}$. Suppose $g\notin G_n^{(1)}$. Then it is sufficient to use the fact that any Laurent polynomial vanishing on a countable set vanishes identically. It follows that, first, $f(1)\neq 0$ for any non-zero L.-p. solution and, second, any two non-zero L.-p. solutions are proportional, in particular to the standard solution $S_\lambda^{(\ell;q)}(y)$.

Instead of the sequence $y=q^{-j}$ one can take $y=q^j\ell^{-n}$ and use the same argument. Note that the above recursive process is the exact analog of the Taylor series expansion around $y=1$ in the nonrelativistic case.

On the other hand, a similar argument works with expansion around 0 or ∞. Substituting in (5.24) the expansion $f(y)=\sum_{k=k_-}^{k_+} f_k y^k$ one obtains $(n+2)$-terms recurrence relation $\sum_{j=0}^{n+1} A_{kj} f_{k-j}=0$ for f_k. The "boundary" coefficients A_{k0} and $A_{k,n+1}$ have simple form

$$A_{k0}=(-1)^n \ell^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} (q^k - q^{\lambda_j} \ell^{1-j}), \tag{5.27a}$$

$$A_{k,n+1}=-\left(\frac{\ell}{q}\right)^{\frac{n(n+1)}{2}} \prod_{j=1}^{n} (q^k - q^{\lambda_j+n+1} \ell^{n-j}). \tag{5.27b}$$

Suppose $g \notin G_n^{(0)}$. Then, since $\ell = q^{-g}$, the coefficient A_{k0} vanishes only for $k=\lambda_1$, and $A_{k,n+1}=0$ only for $k=\lambda_n+n+1$. Hence inevitably $k_-=\lambda_1$, $k_+=\lambda_n$, and the coefficients f_k are determined recursively in a unique way starting from f_{λ_1} or f_{λ_n} which proves the uniqueness of L.-p. solution. ■

The question whether the uniqueness of the L.-p. solution really breaks for $g \in G_n$, remains still open.

It would be interesting to strengthen the above result.

Conjecture 2 *The equation (5.24) with free parameters h_j has a polynomial solution only for h_j given by (3.9) and $\lambda = \{\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n\} \in \mathbb{Z}^n$.*

6. Discussion

The results of the present paper generalize to the case of the Ruijsenaars model and Macdonald polynomials those of [2] obtained for the Calogero-Sutherland model and Jack polynomials. In the nonrelativistic limit $\hbar \to 0$, $g=\text{const}$, the Hamiltonians H_k, operator M, separated polynomials S and equations for them go over into the corresponding objects described in [2].

The crucial element of our approach is the operator identity (4.13) which allows to prove the factorization (4.3) of $MP_\lambda^{(\ell;q)}$ and to establish thus the separation of variables. The identity (4.13) is apparently a quantum analog of the characteristic equation for the classical Lax operator. Moreover, the kernel $\mathcal{M}$ can be considered as a collection of eigenfunctions to the quantized separation variables y_j describing thus the change of basis from 't-representation' to 'y-representation'. Though these analogies with the classical inverse scattering method proved to be useful as an heuristic tool for finding SoV for quantum integrable systems [1], their algebraic/geometric origin is still to be cleared up.

An interesting problem is to search for alternative forms of M. We have presented here two descriptions of M: analytical (4.1) in terms of Askey-Wilson integral, and algebraic (4.31) in terms of the basis $p_{jk\nu}$. Our study of M is based mainly on the

analytical definition. It would be interesting also to develop the theory of M based entirely on the algebraic definition, in particular, to give a purely algebraic proof of the identity (4.13).

When our work was close to be finished we became aware of the preprint [17] of Mangazeev addressing the same problem of SoV for A_2 Macdonald polynomials. His proposal for the operator M is different from ours, using a q-integral rather than a contour integral as we do. Some of his arguments are quite formal, for instance, expressions with the ${}_6\psi_6$-series he is using as a final result are divergent. It seems that our choice of M, compared to that of [17], allows to overcome the problems of convergence of the q-integral and to obtain explicit expressions for M^{-1} and action of M on polynomials. Still, the problem of representing M as a q-integral seems to deserve a further consideration.

Although we can predict the form of the separation polynomial $S_\lambda^{(\ell;q)}(y)$ for the n-particle case and study it in detail (section 5), the corresponding n-particle generalization of the kernel $\mathcal{M}$ is not yet clear, so it is an open problem to separate variables for the A_{n-1} Macdonald polynomials for $n > 3$.

In fact, there are infinitely many "separating" operators $M^{(n)}$, since for any choice of c_λ the operator defined as

$$M^{(n)} : P_\lambda^{(\ell;q)}(t_1, \dots, t_n) \to c_\lambda x^{|\lambda|} \prod_{j=1}^{n-1} S_\lambda^{(\ell;q)}(y_j) \tag{6.1}$$

will serve the purpose. The genuine problem, however, is to choose the coefficients c_λ in such a way that the corresponding kernel $\mathcal{M}^{(n)}$ were given by an explicit expression generalizing (4.2).

Appendix A. Formulas from q-analysis

For reader's convenience, we have collected here the most important definitions and formulas from q-analysis used in the main text. For references see [10, 11, 12, 13]. Especially useful for practical calculations is the collection of formulas in Appendices I and II from [10]. Throughout the text it is assumed that $0 < q < 1$.

The q-shifted factorial and its generalizations are defined as

$$(a;q)_0 := 1, \qquad (a;q)_k := (1-a)(1-aq)\dots(1-aq^{k-1}), \qquad k = 1, 2, \dots\ , \tag{A.1}$$

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad (x;q)_\alpha = \frac{(x;q)_\infty}{(q^\alpha x;q)_\infty}, \qquad \alpha \in \mathbb{C}, \tag{A.2}$$

$$(a_1, a_2, \cdots, a_n;q)_k := (a_1;q)_k(a_2;q)_k \dots (a_n;q)_k, \qquad k = 0, 1, 2, \dots \text{ or } \infty. \tag{A.3}$$

Note the useful relations

$$(qx;q)_\alpha = \frac{1-q^\alpha x}{1-x}(x;q)_\alpha, \qquad (q^{-1}x;q)_\alpha = \frac{1-q^{-1}x}{1-q^{\alpha-1}x}(x;q)_\alpha. \tag{A.4}$$

We make use also of the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \qquad k = 0, 1, \ldots, n, \tag{A.5}$$

q-Gamma and q-Beta functions

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty (1-q)^{z-1}}, \qquad \Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z), \tag{A.6}$$

$$B_q(a,b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)} = (1-q)\frac{(q, q^{a+b};q)_\infty}{(q^a, q^b;q)_\infty}, \tag{A.7}$$

q-integral

$$\int_0^1 d_q t f(t) := (1-q) \sum_{k=0}^{\infty} f(q^k) q^k, \tag{A.8}$$

and basic hypergeometric series ($n \in \mathbb{Z}_{\geq 0}$)

$${}_n\phi_{n-1}\left[\begin{array}{c} a_1, \ldots, a_n \\ b_1, \ldots, b_{n-1} \end{array}; q, y\right] := \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_n; q)_k}{(q, b_1, \ldots, b_{n-1}; q)_k} y^k, \qquad |y| < 1. \tag{A.9}$$

Denoting the expression (A.9) by $f(y)$ we observe that it satisfies the n-th order q-difference equation, see [14] and [12] (section 2.12.3):

$$\left\{ y \prod_{k=1}^{n} (1 - a_k Y) - \prod_{k=1}^{n} (1 - q^{-1} b_k Y) \right\} f(y) = 0 \tag{A.10}$$

where $(Yf)(y) := f(qy)$ and $b_n \equiv q$.

Summation formula for ${}_1\phi_0$ (q-binomial series):

$${}_1\phi_0\left[\begin{array}{c} a \\ - \end{array}; q, y\right] \equiv \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} y^k = \frac{(ay;q)_\infty}{(y;q)_\infty}, \qquad |y| < 1. \tag{A.11}$$

The ϕ_D-type q-Lauricella function [13, 15] of $n-1$ variables x_j is a multi-variable generalization of the basic hypergeometric series:

$$\phi_D\left[\begin{array}{c} a'; b'_1, \ldots, b'_{n-1} \\ c \end{array}; q; x_1, \ldots, x_{n-1}\right] := \sum_{k_1, \ldots, k_{n-1}=0}^{\infty} \frac{(a';q)_{k_1+\ldots+k_{n-1}}}{(c;q)_{k_1+\ldots+k_{n-1}}} \prod_{j=1}^{n-1} \frac{(b'_j;q)_{k_j} x_j^{k_j}}{(q;q)_{k_j}}. \tag{A.12}$$

Andrews [16] has found that ϕ_D can be expressed in terms of the basic hypergeometric function ${}_n\phi_{n-1}$ of one variable:

$$\phi_D\left[\begin{array}{c} a'; b'_1, \ldots, b'_{n-1} \\ c \end{array}; q; x_1, \ldots, x_{n-1}\right] = \frac{(a', b'_1 x_1, \ldots, b'_{n-1} x_{n-1}; q)_\infty}{(c, x_1, \ldots, x_{n-1}; q)_\infty}$$

$$\times\, {}_n\phi_{n-1}\left[\begin{array}{cccc} c/a', & x_1, & \ldots, & x_{n-1} \\ & b'_1 x_1, & \ldots, & b'_{n-1} x_{n-1} \end{array}; q, a'\right]. \tag{A.13}$$

Our main technical tool, on which the proof of the main theorem 3 depends, is the famous Askey-Wilson integral ([10], section 6.1; [11], section 2.6). Let

$$w(a,b,c,d;t) := \frac{(t^2,t^{-2};q)_\infty}{(at,at^{-1},bt,bt^{-1},ct,ct^{-1},dt,dt^{-1};q)_\infty}. \tag{A.14}$$

Then

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{abcd}} \frac{dt}{t}\, w(a,b,c,d;t) = \frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}. \tag{A.15}$$

The cycle Γ_{abcd} depends on parameters a, b, c, d and is defined as follows. Let $C_{z,r}$ be the counter-clockwise oriented circle with the center z and radius r.

If $|a|,|b|,|c|,|d| < 1$ then $\Gamma_{abcd} = C_{0,1}$. The identity (A.15) can be continued analytically for the values of parameters a, b, c, d outside the unit circle provided the cycle Γ_{abcd} is deformed appropriately. In general case

$$\Gamma_{abcd} = C_{0,1} + \sum_{x=a,b,c,d} \sum_{\substack{k\geq 0\\ |x|q^k>1}} (C_{xq^k,\varepsilon} - C_{x^{-1}q^{-k},\varepsilon}), \tag{A.16}$$

ε being small enough for $C_{x^{\pm1}q^{\pm k},\varepsilon}$ to encircle only one pole of the denominator.

The following formulas are useful when studying the classical and nonrelativistic limits of the quantum Ruijsenaars model. Both correspond to $\hbar \to 0$, $q = e^{-\hbar} \to 1$ and differ only in the behaviour of ℓ (3.5). As $q \uparrow 1$,

$$(x;q)_\alpha \to (1-x)^\alpha, \tag{A.17}$$

$$\frac{(q^\alpha;q)_k}{(1-q)^k} \to (\alpha)_k := \alpha(\alpha+1)\dots(\alpha+k-1), \tag{A.18}$$

$${}_n\phi_{n-1}\left[\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_n} \\ q^{\beta_1},\dots,q^{\beta_{n-1}} \end{matrix};q,y\right] \to {}_nF_{n-1}\left[\begin{matrix} \alpha_1,\dots,\alpha_n \\ \beta_1,\dots,\beta_{n-1} \end{matrix};y\right] \tag{A.19}$$

where ${}_nF_{n-1}$ is the standard (generalized) hypergeometric series, and finally (see [9], § 2.5, Corollary 10):

$$\ln(x;q)_\infty = -\hbar^{-1}\,\mathrm{Li}_2(x) + \frac{1}{2}\ln(1-x) + O(\hbar), \qquad x \in (0,1). \tag{A.20}$$

Appendix B. Operator $M_{\alpha\beta}$

In this section the results are collected concerning the two-parametric generalization $M_{\alpha\beta}$ of one-parametric operator family $M \equiv M_{g,2g}$ studied in the main text. It is an open question whether $M_{\alpha\beta}$ provides a SoV for some integrable model.

Let us substitute into the Askey-Wilson integral weight $w(a,b,c,d;t)$ (A.14) the values

$$a = sq^{\frac{\alpha}{2}}, \qquad b = s^{-1}q^{\frac{\alpha}{2}}, \qquad c = rq^{\frac{\beta}{2}}, \qquad d = r^{-1}q^{\frac{\beta}{2}}, \tag{B.1}$$

and introduce the notation (quantum analog of (2.30))

$$\mathcal{L}_q(\nu; x, y) := (\nu xy, \nu xy^{-1}, \nu x^{-1}y, \nu x^{-1}y^{-1}; q)_\infty. \tag{B.2}$$

The kernel

$$\mathcal{K}_{\alpha\beta}(r, s \mid t) := w(sq^{\frac{\alpha}{2}}, s^{-1}q^{\frac{\alpha}{2}}, rq^{\frac{\beta}{2}}, r^{-1}q^{\frac{\beta}{2}}; t) = \frac{(t^2, t^{-2}; q)_\infty}{\mathcal{L}_q\Big(q^{\frac{\alpha}{2}}; s, t\Big)\, \mathcal{L}_q\Big(q^{\frac{\beta}{2}}; r, t\Big)} \tag{B.3}$$

defines the integral operator

$$K_{\alpha\beta} : f(t) \to \frac{1}{2\pi i} \int\limits_{\Gamma^{rs}_{\alpha\beta}} \frac{dt}{t}\, \mathcal{K}_{\alpha\beta}(r, s \mid t) f(t), \tag{B.4}$$

the contour $\Gamma^{rs}_{\alpha\beta}$ being obtained from Γ_{abcd} (A.16) by substitutions (B.1).

Using the Askey-Wilson integral (A.15) we obtain then the formula

$$K_{\alpha\beta} : 1 \to \frac{2B_q(\alpha, \beta)}{(1-q)(q;q)^2_\infty\, \mathcal{L}_q\Big(q^{\frac{\alpha+\beta}{2}}; r, s\Big)}. \tag{B.5}$$

Now we introduce the operator $M_{\alpha\beta}$

$$M_{\alpha\beta} = (K_{\alpha\beta} \cdot 1)^{-1} \circ K_{\alpha\beta}, \tag{B.6}$$

so that $M : 1 \to 1$, and having the kernel

$$\mathcal{M}_{\alpha\beta}(r, s \mid t) = \frac{(1-q)(q;q)^2_\infty (t^2, t^{-2}; q)_\infty\, \mathcal{L}_q\Big(q^{\frac{\alpha+\beta}{2}}; r, s\Big)}{2B_q(\alpha, \beta)\, \mathcal{L}_q\Big(q^{\frac{\alpha}{2}}; s, t\Big)\, \mathcal{L}_q\Big(q^{\frac{\beta}{2}}; r, t\Big)}. \tag{B.7}$$

The kernel $\mathcal{M}$ (4.2) studied in Section 4 is obtained from $\mathcal{M}_{\alpha\beta}$ (B.7) after the substitutions

$$\alpha = g, \quad \beta = 2g, \quad q^{-g} = \ell, \quad r = t_+ = y_+\ell^{\frac{3}{2}}, \quad s = y_-, \quad t = t_-. \tag{B.8}$$

It is natural to think of $M_{\alpha\beta}$ as acting on the space $\mathrm{Ref}(t)$ of reflexive (invariant w.r.t. $t \to t^{-1}$) Laurent polynomials in t. Consider a Laurent polynomial $R^{\alpha\beta}_{j_1j_2k_1k_2} \in \mathrm{Ref}(t)$, $j_{1,2}, k_{1,2} \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} R^{\alpha\beta}_{j_1j_2k_1k_2}(t) \;&:=\; (q^{\frac{\alpha}{2}}st, q^{\frac{\alpha}{2}}st^{-1}; q)_{j_1} (q^{\frac{\alpha}{2}}s^{-1}t, q^{\frac{\alpha}{2}}s^{-1}t^{-1}; q)_{j_2} \\ &\times\; (q^{\frac{\beta}{2}}rt, q^{\frac{\beta}{2}}rt^{-1}; q)_{k_1} (q^{\frac{\beta}{2}}r^{-1}t, q^{\frac{\beta}{2}}r^{-1}t^{-1}; q)_{k_2}. \end{aligned} \tag{B.9}$$

Using the obvious identity

$$\mathcal{K}_{\alpha\beta}(r, s \mid t) R^{\alpha\beta}_{j_1j_2k_1k_2}(t) = \mathcal{K}_{\alpha+j_1+j_2, \beta+k_1+k_2}(rq^{\frac{k_1-k_2}{2}}, sq^{\frac{j_1-j_2}{2}} \mid t), \tag{B.10}$$

together with (B.5), we obtain the formula for action of $M_{\alpha\beta}$ on Laurent polynomials

$$\begin{aligned} M_{\alpha\beta}: R^{\alpha\beta}_{j_1j_2k_1k_2} \quad \to \quad & \frac{(q^\alpha;q)_{j_1+j_2}(q^\beta;q)_{k_1+k_2}}{(q^{\alpha+\beta};q)_{j_1+j_2+k_1+k_2}} \\ & \times(q^{\frac{\alpha+\beta}{2}}rs;q)_{j_1+k_1}(q^{\frac{\alpha+\beta}{2}}rs^{-1};q)_{j_2+k_1} \\ & \times(q^{\frac{\alpha+\beta}{2}}r^{-1}s;q)_{j_1+k_2}(q^{\frac{\alpha+\beta}{2}}r^{-1}s^{-1};q)_{j_2+k_2}. \end{aligned} \tag{B.11}$$

The set of polynomials $R^{\alpha\beta}_{j_1j_2k_1k_2}$ is rich enough to choose from it a basis in $\mathrm{Ref}(t)$, for instance

$$p^\beta_\nu(t) := R^{\alpha\beta}_{00\nu 0}(t) \equiv (q^{\frac{\beta}{2}}rt, q^{\frac{\beta}{2}}rt^{-1})_\nu, \qquad \nu = 0,1,2,\dots. \tag{B.12}$$

More correctly, since $p^\beta_\nu(t) = (-1)^\nu q^{\nu(\nu-1+\beta)/2} r^\nu (t^\nu + t^{-\nu})$+*lower order terms*, $p^\beta_\nu(t)$ is a basis in the space of reflexive Laurent polynomials in variable t with coefficients from $\mathrm{Ref}(r)$. The specialization of the formula (B.11)

$$M_{\alpha\beta}: p^\beta_\nu(t) \to \frac{(q^\beta;q)_\nu}{(q^{\alpha+\beta};q)_\nu} p^{\alpha+\beta}_\nu(s) \tag{B.13}$$

provides thus a tool for calculating explicitely the action of M on any polynomial $\in$ Ref(t).

Analysing the formula (B.13) one obtains the following statement.

Proposition 13 *Let $f \in \mathrm{Ref}(t)$ and suppose f has the parity σ that is $f(-t) = (-1)^\sigma f(t)$. Let $M_{\alpha\beta}: f \to F$. Then: $F \in \mathrm{Ref}(r) \otimes \mathrm{Ref}(s)$, $F(r,s) = F(s,r)|_{\alpha\leftrightarrow\beta}$, $F(-r,-s) = (-1)^\sigma F(r,s)$.*

The integral operator $M_{\alpha\beta}$ simplifies drastically when one of the parameters α, β takes negative integer values.

Proposition 14 *Let $\alpha \in \mathbb{Z}_{\leq 0}$. Then $M_{\alpha\beta}$ turns into the q-difference operator of order $-\alpha$:*

$$M_{\alpha\beta}: f(t) \to \sum_{k=0}^{-\alpha} \xi_k(r,s) f(q^{k+\frac{\alpha}{2}}s) \tag{B.14}$$

where

$$\begin{aligned} \xi_k(r,s) = (-1)^k q^{-\frac{k(k-1)}{2}} & \begin{bmatrix} -\alpha \\ k \end{bmatrix}_q s^{-2k}(1-q^{-\alpha-2k}s^{-2}) \\ \times & \frac{(q^{\frac{\alpha+\beta}{2}}rs, q^{\frac{\alpha+\beta}{2}}r^{-1}s;q)_k (q^{\frac{\alpha+\beta}{2}}rs^{-1}, q^{\frac{\alpha+\beta}{2}}r^{-1}s^{-1};q)_{-\alpha-k}}{(q^{\alpha+\beta};q)_{-\alpha}(q^{-k}s^{-2};q)_{1-\alpha}}. \end{aligned} \tag{B.15}$$

Proof. Instead of analyzing the degeneration of the integral operator defined by the kernel (B.7) it is easier to study the action of $M_{\alpha\beta}$ on the basic polynomials $p^\beta_\nu(t)$ (B.12).

Substituting $f(t) := p_\nu^\beta(t)$ into (B.14) and using (B.13) we obtain, after simplification, the equality

$$\sum_{k=0}^{-\alpha} \xi_k(r,s) \frac{(q^{\frac{\alpha+\beta}{2}+\nu} rs;q)_k (q^{\frac{\alpha+\beta}{2}+\nu} rs^{-1};q)_{-\alpha-k}}{(q^{\frac{\alpha+\beta}{2}} rs;q)_k (q^{\frac{\alpha+\beta}{2}} rs^{-1};q)_{-\alpha-k}} = \frac{(q^{\alpha+\beta+\nu};q)_{-\alpha}}{(q^{\alpha+\beta};q)_{-\alpha}} \tag{B.16}$$

which, after substituting (B.15) and making a series of elementary transformations (see Appendix I to [10]), can be put into the form

$$\sum_{k=0}^{-\alpha} q^{(1-\alpha-\beta-\nu)k} \frac{(q^\alpha, q^{\frac{\alpha+\beta}{2}+\nu} rs, q^{\frac{\alpha+\beta}{2}} r^{-1}s, q^\alpha s^2, q^{1+\frac{\alpha}{2}} s, -q^{1+\frac{\alpha}{2}} s;q)_k}{(q, q^{\frac{\alpha-\beta}{2}+1} rs, q^{\frac{\alpha-\beta}{2}+1-\nu} r^{-1}s, qs^2, q^{\frac{\alpha}{2}} s, -q^{\frac{\alpha}{2}} s;q)_k}$$

$$= \frac{(q^{1-\beta-\nu}, q^{1+\alpha} s^2;q)_{-\alpha}}{(q^{\frac{\alpha-\beta}{2}+1} rs, q^{\frac{\alpha-\beta}{2}+1-\nu} r^{-1}s;q)_{-\alpha}}, \tag{B.17}$$

identical to the summation formula (II.21) from [10]:

$${}_6\phi_5 \left[\begin{matrix} a, qa^{\frac12}, -qa^{\frac12}, b, c, q^\alpha \\ a^{\frac12}, -a^{\frac12}, \frac{aq}{b}, \frac{aq}{c}, aq^{1-\alpha} \end{matrix} ; q, \frac{aq^{1-\alpha}}{bc} \right] = \frac{\left(aq, \frac{aq}{bc};q\right)_{-\alpha}}{\left(\frac{aq}{b}, \frac{aq}{c};q\right)_{-\alpha}} \tag{B.18}$$

for the following identification of the parameters

$$a = q^\alpha s^2, \quad b = q^{\frac{\alpha+\beta}{2}} r^{-1}s, \quad c = q^{\frac{\alpha+\beta}{2}+\nu} rs. \tag{B.19}$$

By the symmetry $\alpha \leftrightarrow \beta$, $r \leftrightarrow s$ the similar statement can be proved for $\beta \in \mathbb{Z}_{\le 0}$. ■

To determine the inversion of $M_{\alpha\beta}$ let us think of r as a parameter and of $M_{\alpha\beta}$ as an operator $M_{\alpha\beta}^r : \mathrm{Ref}(t) \to \mathrm{Ref}(s) : f(t) \to F(s)$. Then, applying $M_{\alpha\beta}^r$ to the basis $p_\nu^\beta(t)$ (B.12) and inverting the formula (B.13), we obtain the following statement.

Proposition 15 *The inversion formula for $M_{\alpha\beta}^r$:*

$$\left(M_{\alpha\beta}^r\right)^{-1} = M_{-\alpha,\alpha+\beta}^r. \tag{B.20}$$

The corresponding kernel is

$$\widetilde{\mathcal{M}}_{\alpha\beta}^r(t \mid s) = \frac{(1-q)(q;q)_\infty^2 (s^2, s^{-2};q)_\infty \, \mathcal{L}_q\left(q^{\frac{\beta}{2}};r,t\right)}{2B_q(-\alpha,\alpha+\beta)\, \mathcal{L}_q\left(q^{-\frac{\alpha}{2}};s,t\right) \mathcal{L}_q\left(q^{\frac{\alpha+\beta}{2}};r,s\right)}. \tag{B.21}$$

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